Linear Programming Duality

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1 Definition

Let $A \in \mathbb{R}^{m \times n}$, $I \subseteq [m]$, $J \subseteq [n]$. Define the linear program $P := \operatorname{LP}(A, b, c, I, J)$ as

$$\min_{x \in \mathbb{R}^n : x_J \geq 0} c^T x \quad \text{where} \quad ((Ax)_i \geq b_i, \forall i \in I) \text{ and } ((Ax)_i = b_i, \forall i \in [m] - I)$$

Then the dual of P is $D := \operatorname{LP}(-A^T, -c, -b, J, I)$, i.e.,

$$\max_{y \in \mathbb{R}^m : y_I \geq 0} b^T y \quad \text{where} \quad ((A^T y)_j \leq c_j, \forall j \in J) \text{ and } ((A^T y)_j = c_j, \forall j \in [n] - J)$$

Lemma 1. *Dual of dual is dual.*

On setting $J = [n]$ and $I = [m]$, we get that the following are duals of each other

$$P : \min_{x \geq 0} c^T x \text{ where } Ax \geq b \qquad\qquad D : \max_{y \geq 0} b^T y \text{ where } A^T y \leq c$$

2 Method for Quick Application

Adapted from Sébastien Lahaie's notes.

1. Express problem in standard form:
   (a) Express as a minimization problem.
   (b) Write each (non-trivial) constraint as $f(x) \leq 0$ or $f(x) = 0$.
   (c) Write each trivial constraint as $x \geq 0$ (i.e., if $x \leq 0$, replace x by $-x$):

   $$\min_{x \in \mathbb{R}^n : x_J \geq 0} c^T x \quad \text{where} \quad (b_i - (Ax)_i \leq 0, \forall i \in I) \text{ and } (b_i - (Ax)_i = 0, \forall i \in [m] - I).$$

2. Add dual variables:
   (a) Create a non-negative dual variable y_i for each inequality constraint $f_i(x) \leq 0$.
   (b) Create an unrestricted dual variable y_i for each equality constraint $f_i(x) = 0$.
   (c) Remove the constraint and add the term $y_i f_i(x)$ to the objective.
   (d) Maximize over dual variables.

   $$\max_{y \in \mathbb{R}^m : y_I \geq 0} \; \min_{x \in \mathbb{R}^n : x_J \geq 0} c^T x + y^T (b - Ax).$$

3. Rearrange terms to express objective as an affine function of primal variables:

   $$\max_{y \in \mathbb{R}^m : y_I \geq 0} \; \min_{x \in \mathbb{R}^n : x_J \geq 0} b^T y + x^T (c - A^T y).$$

4. For each term $x_j g_j(y)$ in the objective, remove the term and add constraint

   (a) $g_j(y) \geq 0$ if x_j is non-negative.
   (b) $g_j(y) = 0$ if x_j is unrestricted.

$$\max_{y \in \mathbb{R}^m : y_J \geq 0} b^T y \quad \text{where} \quad (c_j - (A^T y)_j \geq 0, \forall j \in J) \text{ and } (c_j - (A^T y)_j = 0, \forall j \in [n] - J).$$

5. Rearrange into suitable form

$$\max_{y \in \mathbb{R}^m : y_J \geq 0} b^T y \quad \text{where} \quad ((A^T y)_j \leq c_j, \forall j \in J) \text{ and } ((A^T y)_j = c_j, \forall j \in [n] - J).$$