Integer Programming: Popular Problems

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Definition 1 (Bidirecting a graph)**.** *Given an undirected graph $G := (V, E)$,* bidirection *is the process of replacing each undirected edge $\{u, v\}$ by directed edges (u, v) and (v, u).*

1 Traveling Salesman Problem (TSP)

Definition 2 (TSP)**.** *Let $G := (V, E)$ be a directed graph (if G is undirected, bidirect it). Each edge $e \in E$ has a non-negative cost $c(e)$. Find a simple cycle of length $|V|$ (called tour) having the minimum total cost.*

If $c(u, v) = c(v, u)$ $\forall u \neq v$, then the problem is called symmetric.

Lemma 1 (Hard to approx)**.** *Symmetric TSP is NP-hard to approximate.*

Proof. Let $G := (V, E)$ be an undirected graph. Let α be an arbitrarily large number. Create set $c(u, v) = 1$ if $(u, v) \in E$ and $c(u, v) = (\alpha - 1)n + 2$ otherwise. If G has a Hamiltonian cycle, then the cheapest tour costs n. Otherwise, the cheapest tour costs at least $\alpha n + 1$. If there is an α-approx polynomial-time algorithm for symmetric TSP, there is a polynomial-time algorithm for undirected Hamiltonian cycle. □

Lemma 2 (Subtour elimination IP)**.** *Let x_e be 1 if edge e is in the tour and 0 otherwise. Then this IP finds the min-cost tour*

$$\begin{aligned}
&\min_{x \in \{0,1\}^{|E|}} && c^T x \\
&\textit{where} && \sum_{e \in \delta^{\text{in}}(v)} x_e = 1 \quad \forall v \in V \\
&\textit{and} && \sum_{e \in \delta^{\text{out}}(v)} x_e = 1 \quad \forall v \in V \\
&\textit{and} && \sum_{e \in \delta^{\text{out}}(S)} x_e \geq 1 \quad \forall S \subset V, 2 \leq |S| \leq |V| - 2
\end{aligned}$$

(Note that this IP has $|E|$ binary variables and $2^{|V|} - 2$ constraints.)

Proof. It is easy to see that a tour satisfies this IP. Let x be a solution to the IP. Let $T := \{e \in E : x_e = 1\}$. By the first two constraint families, $\operatorname{indeg}_T(v) = \operatorname{outdeg}_T(v) = 1$ $\forall v \in V$. Hence, T is a collection of disjoint cycles that span V. Let C be one of these cycles. If $|C| \neq |V|$, then $2 \leq |C| \leq |V| - 2$, and the last constraint is violated for $S = C$. Hence, if x is feasible, then T is a tour. Hence, the set of feasible solutions to the IP is also the set of feasible tours. □

Lemma 3 (MIP)**.** *Let $V = [n]$. Let x_e be 1 if edge e is in the tour and 0 otherwise. Then this MIP finds the min-cost tour*

$$\begin{array}{lll} \displaystyle\min_{x\in\{0,1\}^{|E|}, w\in\mathbb{R}^{n-1}} & c^T x & \\ \textit{where} & \displaystyle\sum_{e\in\delta^{\text{in}}(v)} x_e = 1 & \forall v \in V \\ \textit{and} & \displaystyle\sum_{e\in\delta^{\text{out}}(v)} x_e = 1 & \forall v \in V \\ \textit{and} & w_i - w_j + |V|x_{i,j} \le |V| - 1 & \forall (i,j) \in E \cap [n-1]\times[n-1] \end{array}$$

(This MIP has $|E|$ binary variables, $|V| - 1$ real variables, and $\le 2|V| + |E|$ constraints.)

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a valid tour, where $v_n = n$. Let $w_{v_i} = i$. Then the IP is satisfied. Let (x, w) be a feasible solution to the IP. Let $T = \{e \in E : x_e = 1\}$. Then T is a union of disjoint cycles that span V. Suppose there are multiple cycles in T. Let C be a cycle that doesn't contain vertex n. Adding together constraints from the last family for $(i, j) \in C$, we get $|V||C| \le (|V| - 1)|C|$, which is false. Hence, T is a valid tour. □

Lemma 4 (DP)**.** *Let $V = [n]$. Let $f(S, v)$ be the min cost of a simple path (or simple cycle, if $v = n$) whose first vertex is n, last vertex is v, and the other vertices are $S \subseteq [n-1] - \{v\}$. Then*

$$f(S,v) := \begin{cases} 0 & \textit{if } S = \emptyset \textit{ and } v = n \\ c(n,v) & \textit{if } S = \emptyset \textit{ and } v \neq n \\ \displaystyle\min_{u\in S}(f(S-\{u\},u) + c(u,v)) & \textit{if } S \neq \emptyset \end{cases}.$$

The cost of the optimal tour is $f([n-1], n)$. We can compute $f(S, v)$ for all $S \subseteq [n-1]$ and all $v \in [n] - S$ in $\Theta(n^2 2^n)$ time using dynamic programming.

2 Facility Location

Let there be n facilities and m clients. The input is the tuple (c, b, u, h) where

1. $c \in \mathbb{R}^n_{\ge 0}$ and c_j is the cost of opening facility j.
2. $b \in \mathbb{R}^m_{\ge 0}$ and b_i is the demand of client i.
3. $u \in \mathbb{R}^n_{\ge 0}$ and u_j is the capacity of facility j.
4. $h \in \mathbb{R}^{m\times n}_{\ge 0}$ and $h_{i,j}$ is the cost of satisfying unit demand from client i by facility j.

In the capacitated facility location problem, we need to open some facilities and distribute clients' demand onto the open facilities such that the total demand at a facility is at most its capacity and the cost is minimized.

Let $x_j \in \{0, 1\}$ be 1 iff we open facility j. Let $y_{i,j} \in [0, 1]$ be the fraction of client i's demand satisfied by facility j. Then the MIP for this problem is

$$\begin{aligned} \min_{x \in \{0,1\}^n,\, y \in \mathbb{R}_{\geq 0}^{m \times n}} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} h_{i,j} y_{i,j} \\ \text{where} \quad & \sum_{j=1}^{n} y_{i,j} = b_i \qquad \forall i \in [m] \\ \text{and} \quad & \sum_{i=1}^{m} y_{i,j} \leq u_j x_j \quad \forall j \in [n] \end{aligned}$$

Let $B := \sum_{i=1}^{m} b_i$. When $u_j \geq B$ for all j, we say that the problem is uncapacitated. In this case, we can assume without loss of generality that $b_i = 1$ for all i. Hence, the input can be specified using only the pair (c, h), and we have this MIP:

$$\begin{aligned} \min_{x \in \{0,1\}^n,\, y \in \mathbb{R}_{\geq 0}^{m \times n}} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} h_{i,j} y_{i,j} \\ \text{where} \quad & \sum_{j=1}^{n} y_{i,j} = 1 \quad \forall i \in [m] \\ \text{and} \quad & y_{i,j} \leq x_j \qquad \forall i \in [m], \forall j \in [n] \end{aligned}$$

2.1 Reduction from Set Cover

In the set cover problem, given a matrix $A \in \{0,1\}^{m \times n}$ and a vector $c \in \mathbb{R}_{\geq 0}^n$, we need to solve the IP:

$$\min_{x \in \{0,1\}^n} \quad c^T x \quad \text{where} \quad Ax \geq 1.$$

Let $h_{i,j} := \alpha(1 - A[i,j])$, for an arbitrarily large number α. This gives us an uncapacitated facility location instance (c, h). Solve it with x fixed to $\widehat{x}$. Let $(\widehat{x}, \widehat{y})$ be the optimal solution. Let $\beta := c^T \widehat{x} + h^T \widehat{y}$ be the cost.

Lemma 5. *If $\widehat{x}$ is a valid set cover (i.e., $A\widehat{x} \geq 1$), then $\beta = c^T \widehat{x}$. Otherwise, $\beta = c^T \widehat{x} + \alpha$.*

Proof. If $A\widehat{x} \geq 1$, then for all $i \in [m]$, $\exists j_i \in [n]$ such that $A[i, j_i]\widehat{x}_{j_i} = 1$. Let $\widetilde{y}_{i,j}$ be 1 if $j = j_i$ and 0 otherwise. Then $(\widehat{x}, \widetilde{y})$ is a feasible solution of cost $c^T \widehat{x}$.

If $A\widehat{x} \not\geq 1$, then $\exists k \in [m]$ such that $\forall j \in [n]$, $A[k,j]\widehat{x}_j = 0$. Then $\widehat{y}_{k,j} > 0 \implies \widehat{x}_j = 1 \implies A[k,j] = 0$. Hence,

$$\beta \geq c^T \widehat{x} + \sum_{j=1}^{n} h_{k,j} \widehat{y}_{k,j} = c^T \widehat{x} + \alpha \sum_{j : \widehat{y}_{k,j} > 0} \widehat{y}_{k,j} = c^T \widehat{x} + \alpha. \qquad \square$$

Hence, if $\alpha > \|c\|_1$, then $(\widehat{x}, \widehat{y})$ is optimal for this facility location instance iff $\widehat{x}$ is optimal for the set cover instance.

3 Uncapacitated Lot-Sizing

We need to decide on an n-day production plan for a product. Days are numbered from 1 to n. We can choose to open a facility on some days. The facility will open in the morning, produce goods, and close in the evening. Then some of the goods will be sent to the client, and the rest of the goods (if any) will be stored in the warehouse overnight.

1. d_t: client's demand on day t.
2. f_t: cost of opening the facility on day t.
3. p_t: production cost per unit on day t.
4. h_t: cost per unit stored in the warehouse at night on day t.

3.1 MIP Formulation 1

Decision variables:

1. $x_t \in \mathbb{R}_{\geq 0}$: amount produced on day t.
2. $y_t \in \{0, 1\}$: whether facility is open on day t.
3. $s_t \in \mathbb{R}_{\geq 0}$: amount in warehouse at night on day t, for $t \in [n-1]$. $s_0 := 0$, $s_n := 0$.

Let $M_t := \sum_{i=t}^{n} d_t$. MIP 1:

$$\begin{array}{lll} \min\limits_{x \in \mathbb{R}^n_{\geq 0}, y \in \{0,1\}^n, s \in \mathbb{R}^{n-1}_{\geq 0}} & p^T x + h^T s + f^T y & \\ \text{where} & s_{t-1} + x_t = d_t + s_t & \forall t \in [n] \\ \text{and} & x_t \leq M_t y_t & \forall t \in [n] \end{array}$$

Lemma 6. *Consider the LP relaxation of MIP 1 where $y_t \in \{0, 1\}$ is replaced by $y_t \in [0, 1]$. This LP relaxation is not integral.*

Proof. Let $n = 2$. Let $p = 0$, $h = 0$, $d_1 = d_2 = 1$, $f_1 = 2$, $f_2 = 0$. Then every feasible solution to the MIP has cost 2. However, the solution $x_1 = x_2 = 1$, $s = 0$, $y_1 = 1/2$, $y_2 = 1$ is a basic feasible solution and has cost 1. □

3.2 MIP Formulation 2

Decision variables:

1. $z_{i,j} \in \mathbb{R}_{\geq 0}$: amount produced on day i to satisfy demand on day j. When $i > j$, $z_{i,j} := 0$.
2. $y_t \in \{0, 1\}$: whether facility is open on day t.

MIP 2:

$$\begin{array}{lll} \min\limits_{z \in \mathbb{R}^{n^2}_{\geq 0}, y \in \{0,1\}^n} & \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} p_i z_{i,j} + \sum_{t=1}^{n} f_t y_t + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=i}^{j-1} h_k \right) z_{i,j} & \\ \text{where} & \displaystyle\sum_{i=1}^{j} z_{i,j} = d_j \quad \forall j \in [n] & \\ \text{and} & z_{i,j} \leq d_j y_i \qquad \forall 1 \leq i \leq j \leq n & \end{array}$$

Theorem 7. *Consider the LP relaxation of MIP 2 where $y_t \in \{0, 1\}$ is replaced by $y_t \in [0, 1]$. This LP relaxation is integral.*

Proof. Theorem 3 in [1]. □

3.3 DP solution

Lemma 8 (Zero-inventory ordering)**.** *Fix y in MIP 1. Then in any extreme point solution to MIP 1, $x_t > 0 \implies s_{t-1} = 0$.*

Proof. Let $(\widehat{x}, \widehat{s})$ be a solution to MIP 1 (after fixing y), where $\widehat{x}_t > 0$ and $\widehat{s}_{t-1} > 0$. Then $t > 1$ and $y_t = 1$. Let $k := \max_{i=1}^{t-1}(y_i = 1)$.

Let $\delta := \min(\widehat{s}_{t-1}, \widehat{x}_t)$. Then $\delta > 0$. Let $\epsilon \in [-\delta, \delta]$. Define $\widetilde{x}(\epsilon)$ and $\widetilde{s}(\epsilon)$ as

$$\widetilde{x}_i(\epsilon) := \widehat{x}_i + \epsilon \begin{cases} -1 & \text{if } i = k \\ 1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases} \qquad \widetilde{s}_i(\epsilon) := \widehat{s}_i - \epsilon \begin{cases} 1 & \text{if } k \le i < t \\ 0 & \text{otherwise} \end{cases}$$

Then $(\widetilde{x}(\epsilon), \widetilde{s}(\epsilon))$ is also a feasible solution to the MIP. $\widehat{x}$ is the midpoint of $\widetilde{x}(\delta)$ and $\widetilde{x}(-\delta)$, so $\widehat{x}$ is not an extreme point. □

Let $g(i, j)$ be the cost of the min-cost solution for days $[j] - [i-1]$, when $y_i = 1$ and $y_t = 0$ for $t \neq i$. We can compute $g(i, j)$ in $\Theta(j - i)$ time in a straightforward manner. In fact, we can also compute $g(i, j)$ for all $1 \le i \le j \le n$ in $\Theta(n^2)$ time.

Let $f(j)$ be the cost of the min-cost for days $[j]$. Then by Lemma 8,

$$f(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{i=1}^{j}(f(i-1) + g(i, j)) & \text{if } j > 0 \end{cases}.$$

Hence, we can compute $f(j)$ for all j using dynamic programming.

4 Assortment Optimization

There are n products, numbered 1 to n. Product i has revenue r_i and consumer appeal v_i. For a set $S \subseteq [n]$ (called assortment), the probability of a consumer buying product i is

$$p_i(S) := \frac{v_i}{v_0 + \sum_{j \in S} v_j}.$$

Let $C \in \mathbb{Z}_{\ge 1}$ be the assortment capacity. We need to find an assortment $S \subseteq [n]$, where $|S| \le C$, to maximize the expected revenue

$$\sum_{i \in S} r_i p_i(S) = \frac{\sum_{i \in S} r_i v_i}{v_0 + \sum_{i \in S} v_i}.$$

Non-linear IP:

$$\min_{x \in \{0,1\}^n} \quad \frac{\sum_{i=1}^n r_i v_i x_i}{v_0 + \sum_{i=1}^n v_i x_i} \quad \text{where} \quad \sum_{i=1}^n x_i \le C.$$

4.1 Solution using LP

LP:

$$\begin{aligned}
\min_{w_0\in\mathbb{R}, w\in\mathbb{R}^n} \quad & \sum_{i=1}^n r_i w_i \\
\text{where} \quad & w_0 + \sum_{i=1}^n w_i = 1 \\
\text{and} \quad & 0 \le \frac{w_i}{v_i} \le \frac{w_0}{v_0} \qquad \forall i \in [n] \\
\text{and} \quad & \sum_{i=1}^n \frac{w_i}{v_i} \le C\frac{w_0}{v_0}
\end{aligned}$$

Lemma 9 (Extreme points)**.** *In any extreme-point solution to LP, $w_i/v_i \in \{0, w_0/v_0\}$.*

Proof. Let w be an extreme point solution. If $w_0 = 0$, then $w_i = 0$ for all $i \in [n]$, which violates the first constraint. Hence, $w_0 > 0$.

w is tight at at least $n+1$ constraints. So, there can be at most a single value k for which $w_k/v_k \notin \{0, w_0/v_0\}$. Assume such a k exists. Then the last constraint is tight. Hence, for $S := \{i : w_i/v_i = w_0/v_0\}$, we get

$$\frac{w_k}{v_k} = C\frac{w_0}{v_0} - \sum_{i\neq k}\frac{w_i}{v_i} = (C - |S|)\frac{w_0}{v_0}.$$

Since $C - |S| \in \mathbb{Z}$, we get that $w_k/v_k \in \{0, w_0/v_0\}$. This is a contradiction. Hence, no such k exists. □

Lemma 10 (LP vs IP)**.** *The optimal objective value of the IP and LP are the same. Furthermore, we can solve the IP using an extreme-point solution to the LP.*

Proof. Let w be an optimal extreme-point solution to the LP. Then $x_i := w_i v_0/v_i w_0$ is feasible for the IP.

Let x be an optimal solution to the IP. Let $\alpha := v_0 + \sum_{i=1}^n v_i x_i$. Let $w_0 := v_0/\alpha$ and for $i \in [n]$, let $w_i := v_i x_i/\alpha$. Then w is feasible for the LP. □

References

[1] Imre Barany, Tony Van Roy, and Laurence A Wolsey. Uncapacitated lot-sizing: The convex hull of solutions. In *Mathematical Programming at Oberwolfach II*, pages 32–43. Springer, 1984. doi:10.1007/BFb0121006.