Integer Programming: Cutting Plane Method

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Lemma 1 (Disjunction). *Let $S_1, S_2 \subseteq \mathbb{R}^n$. If $a^T x \le \alpha$ for all $x \in S_1$, and $b^T x \le \beta$ for all $x \in S_2$, then $\min(a,b)^T x \le \max(\alpha, \beta)$ for all $x \in S_1 \cup S_2$. ($\min(a,b)_i := \min(a_i, b_i)$.)*

Definition 1 (Inequality dominance). *Let $S \subseteq \mathbb{R}^n$. The inequality $a^T x \ge b$ dominates $c^T x \ge d$ in set S if $\{x \in S : a^T x \ge b\} \subseteq \{x \in S : c^T x \ge d\}$.*

Definition 2 (Mixing). *Suppose we are given a set S of inequalities $\{a_i^T x \ge b_i : i \in I\} \cup \{a_i^T x = b_i : i \in [m] - I\}$. Let $w \in \mathbb{R}^m$, where $w_i \ge 0$ for $i \in I$. Then the inequality $(w^T A)x \ge w^T b$ is called a* mixing *of the inequalities S (we are essentially doing a linear combination of the inequalities to get a new inequality).*

Theorem 2 (Valid dominated by mixing). *Let $A \in \mathbb{R}^{m \times n}$, $I \subseteq [m]$, and $J \subseteq [n]$. Let $P := \{x \in \mathbb{R}^n : (x_j \ge 0, \forall j \in J), ((Ax)_i \ge b_i, \forall i \in I), ((Ax)_i = b_i, \forall i \in [m] - I)\}$ be a non-empty polyhedron. Let $c^T x \ge \gamma$ be an inequality satisfied by all points in P. Let $Q := \{w \in \mathbb{R}^m : (y_i \ge 0, \forall i \in I), ((A^T w)_j \le c_j, \forall j \in J), ((A^T w)_j = c_j, \forall j \in [n] - J)\}$. Then $Q \ne \emptyset$. Let $\widehat{w} \in \operatorname{argmax}_{w \in Q} b^T w$. Then $c^T x \ge \gamma$ is dominated by $(\widehat{w}^T A)x \ge \widehat{w}^T b$ in $S := \{x \in \mathbb{R}^n : x_j \ge 0, \forall j \in J\}$.*

Proof. Without loss of generality, assume $\{x \in P : c^T x = \gamma\} \ne \emptyset$, since we can increase γ. Then γ is the optimal objective value of the LP $\min_{x \in P} c^T x$. The dual of this LP is $\max_{w \in Q} b^T w$. By strong duality, $Q \ne \emptyset$ and $b^T \widehat{w} = \gamma$.

Let $\widehat{x} \in \{x \in S : \widehat{w}^T A x \ge \widehat{w}^T b\}$. We will show that $c^T \widehat{x} \ge \gamma$, which will imply that $(\widehat{w}^T A)x \ge \widehat{w}^T b$ dominates $c^T x \ge \gamma$. When $j \in J$, then $\widehat{x}_j \ge 0$ and $(A^T \widehat{w})_j \le c_j$, so $\widehat{x}_j (c - A^T \widehat{w})_j \ge 0$. When $j \notin J$, then $(A^T \widehat{w})_j = c_j$, so $\widehat{x}_j (c - A^T \widehat{w})_j = 0$. Hence, $\widehat{x}^T (c - A^T \widehat{w}) \ge 0$, so $c^T \widehat{x} \ge \widehat{w}^T A \widehat{x} \ge \widehat{w}^T b = \gamma$. □

Lemma 3 (Rounding). *Let $S \subseteq \mathbb{R}^n$ and $a \in \mathbb{Z}^n$. Then*

$$(\forall x \in S, a^T x \ge b) \implies (\forall x \in S \cap \mathbb{Z}^n, a^T x \ge \lceil b \rceil).$$

$$(\forall x \in S, a^T x \le b) \implies (\forall x \in S \cap \mathbb{Z}^n, a^T x \le \lfloor b \rfloor).$$

This transformation is called rounding.

Corollary 3.1. *Let $S \subseteq \mathbb{R}^n_{\ge 0}$. Then*

$$(\forall x \in S, a^T x \ge b) \implies (\forall x \in S \cap \mathbb{Z}^n, \lceil a \rceil^T x \ge \lceil b \rceil).$$

$$(\forall x \in S, a^T x \le b) \implies (\forall x \in S \cap \mathbb{Z}^n, \lfloor a \rfloor^T x \le \lfloor b \rfloor).$$

1 Chvátal-Gomory Process

Definition 3 (Chvátal-Gomory process). *Let $P \subseteq \mathbb{R}^n$ be a polyhedron described as a set S of inequalities. Assign a* depth *of 0 to all inequalities in S. Repeated application of the following operations (for any number of iterations) is called the Chvátal-Gomory (CG) process:*

1. *Mix a subset S' of inequalities from S to get a new inequality $c^T x \geq d$ such that $c \in \mathbb{Z}^n$.*
2. *Add the inequality $c^T x \geq \lceil d \rceil$ to S. Assign the depth $1 + \max_{i \in S'} \text{depth}(i)$ to this newly-added inequality.*

The output of the process is S, the set of (original and newly-added) inequalities.

Lemma 4. *Let Q be the output of some CG process on polyhedron P. Let $P_I := \text{convexHull}(P \cap \mathbb{Z}^n)$. Then $P_I \subseteq Q \subseteq P$.*

Definition 4 (CG rank and CG closure). *Let $P \subseteq \mathbb{R}^n$ be a polyhedron, represented as a set S of inequalities. For any inequality $c^T d \geq d$, where $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$, the CG rank of the inequality is r if it can be obtained as a depth r inequality in some CG process.*

The r^{th} CG-closure of P is defined as the set of inequalities (and the associated polyhedron) of CG-rank at most r.

The CG-rank of a polyhedron is the smallest number t such that the t^{th} CG-closure of P is $\text{convexHull}(P \cap \mathbb{Z}^n)$ (and ∞ if no such integer t exists).

Lemma 5. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron, $P_I := \text{convexHull}(P \cap \mathbb{Z}^n)$, and $P^{(r)}$ be the r^{th} CG closure of P. Then $P_I \subseteq P^{(r)} \subseteq P$.*

Lemma 6. *Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Let $P' := \{x : Ax \leq b, (w^T A)x \leq \lfloor w^T b \rfloor \, \forall w \in [0,1]^m$ such that $w^T A \in \mathbb{Z}^n\}$. Then P' is the first CG-closure of P.*

Proof sketch. Any rank 1 inequality can be written as $(w^T A)x \leq \lfloor w^T b \rfloor$ for some $w \geq 0$, where $w^T A \in \mathbb{Z}^n$. Let $u := w - \lfloor w \rfloor$. We can show that $(u^T A)x \leq \lfloor u^T b \rfloor$ is a rank-1 CG inequality, and $w^T Ax \leq \lfloor w^T b \rfloor$ is implied by $(u^T A)x \leq \lfloor u^T b \rfloor$ and $Ax \leq b$. □

Theorem 7. *If P is a rational polyhedron, then the first CG-closure of P is also a rational polyhedron.*

Theorem 8 (Chvátal-Gomory). *Given a rational polyhedron P described as a set S of inequalities, we can obtain $\text{convexHull}(P \cap \mathbb{Z}^n)$ using the CG process for a finite number of iterations (by appropriately choosing which inequalities to mix in each step).*

Corollary 8.1. *The CG rank of a rational polyhedron is finite.*

1.1 Extra Results

Theorem 9. *There exist polyhedra whose CG rank is super-polynomial in mn.*

Theorem 10. *Polyhedra in $[0,1]^n$ have CG rank at most $n^2(1 + \log n)$. There exist polyhedra in $[0,1]^n$ whose CG rank is at least cn^2 for some constant c.*

2 Examples

Theorem 11 (Non-bipartite Matching)**.** *Let $G := (V, E)$ be an undirected graph. Let*

$$P := \left\{ x \in \mathbb{R}^{|E|} : \left(\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in V \right), (x_e \geq 0, \forall e \in E) \right\}.$$

Then for any $S \subseteq V$ where $|S|$ is odd, the following inequality has CG rank at most 1:

$$\sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2}. \qquad \text{(here } E(S) := \{(u, v) \in E : u \in S, v \in S\}\text{)}$$

Proof.

$$\sum_{v \in S} \frac{1}{2} \left(\sum_{e \in \delta(v)} x_e \leq 1 \right) + \sum_{e \in \delta(S)} \frac{1}{2} (-x_e \leq 0) = \left(\sum_{e \in E(S)} x_e \leq \frac{|S|}{2} \right).$$

Then round this inequality. $\lfloor |S|/2 \rfloor = (|S| - 1)/2$. □

Theorem 12 (Independent set)**.** *Let $G := (V, E)$ be an undirected graph. Let*

$$P := \left\{ x \in \mathbb{R}^{|V|} : (x_u + x_v \leq 1, \forall (u, v) \in E), (0 \leq x_v \leq 1, \forall v \in V) \right\}.$$

Then for any odd-cycle C, the following is a CG rank 1 inequality:

$$\sum_{v \in C} x_v \leq \frac{|C| - 1}{2}. \tag{1}$$

For any clique $S \subseteq V$, the following is an inequality of CG rank at most $\lceil \log_2(|S| - 1) \rceil$:

$$\sum_{v \in S} x_v \leq 1. \tag{2}$$

Proof. Equation (1) has rank at most 1 because

$$\sum_{(u,v) \in C} \frac{1}{2} (x_u + x_v \leq 1) = \left(\sum_{v \in C} x_v \leq \frac{|C|}{2} \right).$$

It has rank exactly 1 since it contains $|C| \geq 3$ terms, but the original inequalities of P contain at most 2 terms.

WLoG, assume $S := \{1, 2, \ldots, n\}$. Let $m := \lceil \log_2(n - 1) \rceil$. For $0 \leq i \leq m$, let $r_i := \min(n, 2^i + 1)$. Then $r_{m-1} < n = r_m$. Let $B(k)$ be the proposition that for any $R \subseteq S$ such that $|R| \leq r_k$, $\sum_{i \in R} x_i \leq 1$ is a CG inequality of rank at most k. Then $B(m)$ would imply Eq. (2).

We will show $B(k)$ by induction on k. If $|R| \leq 2 = r_0$, then $\sum_{i \in R} x_i \leq 1$ is a constraint of P (since S is a clique), and so has rank 0. Now let $r_{k-1} < |R| \leq r_k$ for some $k \geq 1$.

By the inductive hypothesis, for any $T \subseteq S$ such that $|T| = r_{k-1}$, $\sum_{i \in T} x_i \leq 1$ is an inequality of CG rank at most $k-1$. Then

$$\sum_{T \subseteq R : |T| = r_{k-1}} \left(\sum_{i \in T} x_i \leq 1 \right) = \left(\binom{|R|-1}{r_{k-1}-1} \sum_{i \in R} x_i \leq \binom{|R|}{r_{k-1}} \right).$$

$$\binom{|R|}{r_{k-1}} = \frac{|R|}{r_{k-1}} \binom{|R|-1}{r_{k-1}-1}, \qquad \frac{|R|}{r_{k-1}} \leq \frac{r_k}{r_{k-1}} = 2 - \frac{1}{r_{k-1}}.$$

Hence, $\sum_{i \in R} x_i \leq 1$ is an inequality of CG rank at most k. By induction, $B(k)\ \forall k$. □

Theorem 13 (Knapsack, [1]). *Let $P := \{x \in [0,1]^n : a^T x \leq b\}$, where $b \in \mathbb{Z}_{\geq 1}$, and $a \in ([1,b] \cap \mathbb{Z})^n$. For any $S \subseteq [n]$, let $a(S) := \sum_{i \in S} a_i$. Let $C \subseteq [n]$ be a* minimal cover, *i.e., $a(C) > b$ and $a(C') \leq b$ for all $C' \subsetneq C$. Then $\sum_{i \in C} x_i \leq |C| - 1$ is an inequality of CG rank at most 1.*

Proof.

$$\begin{aligned} &\sum_{i \in C} \left(1 - \frac{a_i}{b+1}\right)(x_i \leq 1) + \sum_{i \in [n]-C} \left(\frac{a_i}{b+1}\right)(-x_i \leq 0) + \frac{(a^T x \leq b)}{b+1} \\ &= \left(\sum_{i \in C} x_i \leq |C| - \frac{a(C)-b}{b+1} \right). \end{aligned}$$

By minimality of C, we get $b < a(C) \leq 2b$, so $(a(C)-b)/(b+1) \in (0,1)$. □

References

[1] B.L. Dietrich and L.F. Escudero. Obtaining clique, cover and coefficient reduction inequalities as Chvatal-Gomory inequalities and Gomory fractional cuts. *European Journal of Operational Research*, 73(3):539–546, 1994. doi:10.1016/0377-2217(94)90250-X.