Integer Programming: Total Unimodularity

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Based on lecture notes by Prof. Karthik (Lecture 10) and Prof. Eteasmi.

1 Definition and Motivation

Definition 1 (Integral matrix). *$A \in \mathbb{R}^{m \times n}$ is* integral *iff each entry in A is an integer.*

Definition 2 (Total Unimodularity). *A matrix $A \in \mathbb{R}^{m \times n}$ is* totally unimodular *(TU) iff for every square submatrix B of A, we have $\det(B) \in \{-1, 0, 1\}$.*

Lemma 1 (Integral inverse). *Let A be TU. Then for every square submatrix B of A, B^{-1} is integral.*

Proof sketch. If A is TU, then B is also TU. Let $B \in \mathbb{R}^{n \times n}$. Then

$$(B^{-1})[i, j] = \frac{(-1)^{i+j} \det(C_{i,j})}{\det(B)} \quad \text{where} \quad C_{i,j} = B[[n] - \{j\}, [n] - \{i\}].$$

$\det(B), \det(C_{i,j}) \in \{-1, 0, 1\}$ because B is TU. □

Theorem 2 (TU polyhedron). *Let $P := \{x \in \mathbb{R}^n : (a_i^T x = b_i, \forall i \in E) \wedge (a_i^T x \geq b_i, \forall i \in I)\}$ be a non-empty polyhedron, where $b_i \in \mathbb{Z}$ for all $i \in I \cup E$. Let A be a matrix whose rows are $\{a_i^T : i \in I \cup E\}$. If A is TU, then P is integral.*

Proof sketch. We need to show that every minimal face of P contains an integral vector. Every minimal face F is given by $\{x : Bx = c\}$, which is a subsystem of $Ax = b$. Find a basis U of the columns of B. Then U would be a full-rank square submatrix of B. Use $U^{-1}c$ to construct an integral point in F. □

Theorem 3 (Hoffman-Kruskal). *Let $A \in \mathbb{Z}^{m \times n}$. A is TU iff $\{x : Ax \leq b\}$ is integral for all $b \in \mathbb{Z}^m$.*

2 TUity-Preserving Operations on Matrices

Lemma 4. *Let $A \in \mathbb{R}^{m \times n}$.*

1. *A is TU iff $-A$ is TU.*
2. *A is TU iff A^T is TU.*
3. *If B is obtained by rearranging the rows or columns of A, then A is TU iff B is TU.*

4. *If B is obtained by multiplying a row or column of A by a scalar $\alpha \in \{-1, 0, 1\}$, then A is TU $\implies$ B is TU.*
5. *A is TU iff $[A, I]$ is TU.*
6. *If A' is obtained by pivoting A at (i, j), then A' is TU if A is TU.*
7. *If A is invertible, then A is TU iff A^{-1} is TU.*

1, 2, 3, 4 are trivial to prove. 7 is a corollary of 5 and 6, since we can obtain $[I, A^{-1}]$ by repeatedly pivoting $[A, I]$.

Proof sketch of 5. For any square submatrix containing a few rows and columns from I, repeatedly pivot on elements of I till we get a submatrix of A. □

Proof of 6. Let $J \subseteq [m]$ and $K \subseteq [n]$. Let $B := A[J, K]$ and $B' := A'[J, K]$. Then $\det(B) \in \{-1, 0, 1\}$ because A is TU. We will show that $\det(B') \in \{-1, 0, 1\}$.

If $i \in J$, then B' can be obtained by performing row operations on B. Hence, $\det(B') = \det(B) \in \{-1, 0, 1\}$. If $i \notin J$ and $j \in K$, then B' has a zero column, so $\det(B') = 0$.

Suppose $i \notin J$ and $j \notin K$. Let $J' := \{i\} \cup J$ and $K' := \{j\} \cup K$. Let $C := A[J', K']$ and $C' := A'[J', K']$. Then C' can be obtained by performing row operations on C. Hence, $\det(C') = \det(C) \in \{-1, 0, 1\}$. Also,

$$C' = \begin{bmatrix} 1 & A'[i, K] \\ \mathbf{0} & B' \end{bmatrix}.$$

Hence, $\det(C') = \det(B')$. Hence, $\det(B') \in \{-1, 0, 1\}$. □

3 Conditions for TUity

Lemma 5 (Sufficient condition)**.** *Let $A \in \{-1, 0, 1\}^{m \times n}$. Then A is TU if each column of A contains at most two non-0 elements and $\exists M \subseteq [m]$ (subset of rows) such that every column j with two non-0 entries satisfies*

$$\sum_{i \in M} A[i, j] = \sum_{i \notin M} A[i, j]. \tag{1}$$

Proof sketch. Let B be the smallest submatrix of A such that $\det(B) \notin \{-1, 0, 1\}$. Every column of B has exactly two non-0 elements, else we can construct a smaller counterexample. Equation (1) implies that rows of B are linearly dependent, and so $\det(B) = 0$. □

Lemma 6 (Characterization)**.** *Let $A \in \{-1, 0, 1\}^{m \times n}$. A is TU iff $\forall J \subseteq [m]$, $\exists K \subseteq J$,*

$$\left| \sum_{i \in K} A[i, j] - \sum_{i \in J - K} A[i, j] \right| \leq 1 \qquad \forall j \in [n].$$

4 Examples

Lemma 7 (Interval matrix)**.** *A matrix $A \in \{0, 1\}^{m \times n}$ is called an* interval matrix *if in each column, all ones are in consecutive positions. An interval matrix is TU.*

Proof sketch. Use Lemma 6 with K as alternate rows of J. $\square$

Lemma 8 (Directed incidence matrix)**.** *Let $G := (V, E)$ be a directed graph. The incidence matrix of G is defined as the matrix $A \in \{-1, 0, 1\}^{|V| \times |E|}$, where*

$$A[w, (u, v)] := \begin{cases} 0 & \text{if } w \notin \{u, v\} \\ -1 & \text{if } w = u \\ 1 & \text{if } w = v \end{cases}.$$

Then A is TU.

Proof sketch. Use Lemma 5 with $M = \emptyset$. $\square$

Lemma 9. *Let $A \in \{0, 1\}^{n \times n}$, where $A[i, j] = 1$ iff $j \in \{i + 1, i + 1 - n\}$. Then $\det(A)$ is 0 if n is even and 2 if n is odd.*

Proof sketch. Apply two row operations to the determinant to reduce to an $(n-2)\times(n-2)$ matrix of the same structure. $\square$

Lemma 10 (Undirected incidence matrix)**.** *Let $G := (V, E)$ be an undirected graph. The incidence matrix of G is defined as the matrix $A \in \{0, 1\}^{|V| \times |E|}$, where $A[v, e]$ is 1 iff v is an endpoint of e. Then A is TU iff G is bipartite.*

Proof sketch. If G is bipartite with vertex partitions L and R, use Lemma 5 with $M = L$. If G is not bipartite, it contains a cycle C of odd length. Let V' and E' be the vertices and edges in C. Then $\det(A[V', E']) = 2$, by Lemma 9. $\square$