Integer Programming: Criteria for Integrality

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Content in this section is based on Prof. Karthik's notes: Lecture 9.

Definition 1 (Rational Polyhedra)**.** *A polyhedron* $P \subseteq \mathbb{R}^n$ *is rational if it can be expressed as* $P = \{x : (a_i^T x = b_i, \forall i \in E) \wedge (a_i^T x \geq b_i, \forall i \in I)\}$, *where* $a_i \in \mathbb{Q}^n$ *and* $b_i \in \mathbb{Q}$.

Theorem 1. *Let* $P \subseteq \mathbb{R}^n$ *be a rational polyhedron. For any set* $S \in \mathbb{R}^n$, *define the predicate* $\operatorname{hasInt}(S) : S \cap \mathbb{Z}^n \neq \emptyset$. *Then the following are equivalent.*

1. $P = \operatorname{convexHull}(P \cap \mathbb{Z}^n)$.
2. *For every face* F *of* P, $\operatorname{hasInt}(F)$.
3. *For every minimal face* F *of* P, $\operatorname{hasInt}(F)$.
4. $\forall c \in \mathbb{R}^n$, $\max_{x \in P} c^T x \neq \infty \implies \operatorname{hasInt}\left(\operatorname{argmax}_{x \in P} c^T x\right)$.
5. $\forall c \in \mathbb{Z}^n$, $\max_{x \in P} c^T x \neq \infty \implies \operatorname{hasInt}\left(\operatorname{argmax}_{x \in P} c^T x\right)$.
6. $\forall c \in \mathbb{Z}^n$, $\max_{x \in P} c^T x \neq \infty \implies \max_{x \in P} c^T x \in \mathbb{Z}$.