Weighted Darboux Integrals

Eklavya Sharma

We modify the theory of Darboux integration to incorporate *weights*. This is similar to how the Riemann-Stieltjes integral is obtained by incorporating weights in the definition of Riemann integral.

1 Preliminaries

We assume that the domain of all functions is non-empty, i.e., if we write $f : X \to Y$, then we assume $X \neq \emptyset$.

Definition 1 (monotonic function). *A function $f : [a, b] \to \mathbb{R}$ is* monotonic *if $f(y) \leq f(z)$ for all $a \leq y < z \leq b$.*

1.1 Supremum and Infimum

For a set $X \subseteq \mathbb{R}$, if X does not have an upper bound, then $\sup(X) := \infty$, and if X does not have a lower bound, then $\inf(X) := -\infty$.

Lemma 1. *For any $f, g : X \to \mathbb{R}$,*

$$\sup_{x\in X}(f(x) + g(x)) \leq \sup_{x\in X} f(x) + \sup_{x\in X} g(x), \quad \inf_{x\in X}(f(x) + g(x)) \geq \inf_{x\in X} f(x) + \inf_{x\in X} g(x).$$

Lemma 2. *For any $f : X \to \mathbb{R}$ and $z \in \mathbb{R} - \{0\}$,*

$$\sup_{x\in X} zf(x) = \begin{cases} z \sup_{x\in X} f(x) & \text{if } z > 0 \\ z \inf_{x\in X} f(x) & \text{if } z < 0 \end{cases}.$$

Lemma 3. *Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then*

$$\sup_{x\in X} f(x) + \sup_{y\in Y} g(y) = \sup_{x\in X, y\in Y}(f(x) + g(y)).$$

$$\inf_{x\in X} f(x) + \inf_{y\in Y} g(y) = \inf_{x\in X, y\in Y}(f(x) + g(y)).$$

Definition 2 (monotonic limits). *Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. For $c \in (a, b]$, define $f^-(c) := \sup_{x\in(a,c)} f(x)$. For $c \in [a, b)$, define $f^+(c) := \inf_{x\in(c,b)} f(x)$.*

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. Then for any $c \in [a, b)$, we have $(f^+)^+(c) = (f^-)^+(c) = f^+(c)$, and for any $c \in (a, b]$, we have $(f^-)^-(c) = (f^+)^-(c) = f^-(c)$.*

Proof. Let $g(x) := f^+(x)$ for $x \in [a, b)$. Pick any $\epsilon > 0$. $f^+(c) = \inf_{x\in(c,b)} f(x)$. Hence, $\exists z \in (c, b)$ such that $f(z) < f^+(c)+\epsilon$. Now $g(c) \le g^+(c) \le g((c+z)/2) = f^+((c+z)/2) \le f(z) < f^+(c) + \epsilon = g(c) + \epsilon$. Since this is true for all $\epsilon > 0$, we get $g^+(c) = g(c)$, so $(f^+)^+(c) = f^+(c)$.

Now let $g(x) := f^-(x)$ for all $x \in (a, b]$. $g^+(c) = \inf_{x\in(c,b)} g(x) \le \inf_{x\in(c,b)} f(x) = f^+(c)$. $g^+(c) = \inf_{x\in(c,b)} g(x) = \inf_{x\in(c,b)} f^-(x) \ge \inf_{x\in(c,b)} f((c+x)/2) = \inf_{x\in(c,(c+b)/2)} f(x) = f^+(c)$. Hence, $g^+(c) = f^+(c)$, so $(f^-)^+(c) = f^+(c)$.

We can similarly show that for any $c \in (a, b]$, we have $(f^-)^-(c) = (f^+)^-(c) = f^-(c)$. □

1.2 Splitting Intervals

Lemma 5. *Let $W : [a, b] \to \mathbb{R}$ be monotonic and $a < b$. Let $0 < \delta < 1$ and $\mu := (1 - \delta)W^+(a) + \delta W^-(b)$. Then $\exists c \in (a, b)$ such that $W^-(c) \le \mu$ and $W^+(c) \ge \mu$.*

Proof. For all $x \in (a, b)$, we have $W^+(a) \le W(x) \le W^-(b)$. If $W^+(a) = W^-(b)$, then $W(x) = W^+(a) = W^-(b) = \mu$ for all $x \in (a, b)$. Hence, for $c = (a+b)/2$, we get $W^-(c) = W(c) = W^+(c) = \mu$. Now assume $W^+(a) < W^-(b)$. Then $W^+(a) < \mu < W^-(b)$.

Let $A := \{z \in (a, b) : W(z) < \mu\}$ and $B := \{z \in (a, b) : W(z) \ge \mu\}$. If A is empty, then $W(z) \ge \mu > W^+(a)$ for all $z \in (a, b)$, which contradicts the fact that $W^+(a) = \inf_{x\in(a,b)} W(x)$. Hence, $A \ne \emptyset$. Similarly, $B \ne \emptyset$.

Let $c := \inf(B)$. Then $(a, c) \subseteq A$ and $(c, b) \subseteq B$. Hence, $W^-(c) = \sup_{x\in(a,c)} W(x) \le \mu$ and $W^+(c) = \inf_{x\in(c,b)} W(x) \ge \mu$. □

Lemma 6. *Let $W : [a, b] \to \mathbb{R}$ be monotonic and $a < b$. Let $0 < \delta < 1$ and $\mu := (1 - \delta)W^+(a) + \delta W^-(b)$. Then $\exists y, z \in (a, b)$ such that $W(y) \le \mu$ and $W(z) \ge \mu$.*

Proof. By Theorem 5, we get $c \in (a, b)$ such that $W^-(c) \le \mu$ and $W^+(c) \ge \mu$. Pick $y := (a + c)/2$ and $z := (c + b)/2$. Then $\mu \ge W^-(c) = \sup_{x\in(a,c)} W(x) \ge W(y)$ and $\mu \le W^+(c) = \inf_{x\in(c,b)} W(x) \le W(z)$. □

Lemma 7. *Let $W_1, \dots, W_k : [a, b] \to \mathbb{R}$ be monotonic and $a < b$. Then $\exists P^* = (x_0, \dots, x_{k+1}) \in \mathcal{P}(a, b)$ such that $W_j^-(x_i) - W_j^+(x_{i-1}) \le (W_j^-(b) - W_j^+(a))/2$ for all $j \in [k + 1]$.*

Proof sketch. For any partition $P = (y_0, \dots, y_n) \in \mathcal{P}(a, b)$ and any $j \in [k]$, there exists at most one $i \in [n]$ such that $W_j^-(y_i) - W_j^+(y_{i-1}) > (W_j^-(b) - W_j^+(a))/2$. We call such an i a W_j-long index.

Let $P_0 = (a, b) \in \mathcal{P}(a, b)$. For all $j \in [k]$, construct P_j by finding a W_j-long index in P_{j-1} (if it exists) and splitting it into two using Theorem 5. If a W_j-long index doesn't exist, then set $P_j = P_{j-1}$. Then P_k is our P^*. □

Lemma 8. *Let $W_1, \dots, W_k : [a, b] \to \mathbb{R}$ be monotonic functions and $a < b$. Then for any $0 < \epsilon < 1$ there exists $P = (x_0, \dots, x_n) \in \mathcal{P}(a, b)$ such that $n \le (k + 1)^{\lceil \log_2(1/\epsilon)\rceil} \le (k + 1)(1/\epsilon)^{\log_2(k+1)}$ and for all $i \in [n]$ and $j \in [k]$, we have $W_j^-(x_i) - W_j^+(x_{i-1}) \le \epsilon(W_j^-(b) - W_j^+(a))$.*

Proof sketch. Set $P_0 = (a, b) \in \mathcal{P}(a, b)$. For any $t \in \mathbb{N}$, we now show how to construct P_t using P_{t-1}. Let $P_{t-1} = (y_0, \dots, y_m)$. For each $i \in [n]$, split each interval $[y_{i-1}, y_i]$

into a subpartition using Theorem 7. This gives us P_t, and P_t is a sequence of at most $1+m(k+1)$ numbers. Hence, if $P_t = (x_0, \ldots, x_n)$, then $n \leq (k+1)^t$ and for every $i \in [n]$ and $j \in [k]$, we get $W^-(x_i) - W^+(x_{i-1}) \leq 2^{-t}(W^-(b) - W^+(a))$. If we set $t = \lceil \log_2(1/\epsilon) \rceil$, then $2^{-t} \leq \epsilon$. □

1.3 Uniform Continuity

Definition 3 (Uniform Continuity)**.** *For $X \subseteq \mathbb{R}$, the function $f : X \to \mathbb{R}$ is* uniformly continuous *if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X$, if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq \epsilon$.*

Definition 4. *Let $\gamma \in \mathbb{R}_{\geq 0}$, $X \subseteq \mathbb{R}$, and $f : X \to \mathbb{R}$. f is γ-*lipschitz *if $|f(x) - f(y)| \leq \gamma|x - y|$ for all $x, y \in X$.*

Lemma 9. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. The following are equivalent:*

1. *$f^-(x) = f(x)$ for all $x \in (a, b]$ and $f^+(x) = f(x)$ for all $x \in [a, b)$.*
2. *f is uniformly continuous on $[a, b]$.*

Proof. Suppose f is uniformly continuous.

Pick any $z \in (a, b]$. Pick any $\epsilon > 0$. By f's uniform continuity, $\exists 0 < \delta < z - a$ such that for all $a \leq x < y \leq b$, $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \epsilon$. Then $f(z) - f(z - \delta) \leq \epsilon$. Hence,

$$f(z) \geq f^-(z) = \sup_{y \in (a,z)} f(y) \geq f(z - \delta) \geq f(z) - \epsilon.$$

Since this holds for all $\epsilon > 0$, we get $f^-(z) = f(z)$.

Pick any $z \in [a, b)$. Pick any $\epsilon > 0$. By f's uniform continuity, $\exists 0 < \delta < b - z$ such that for all $a \leq x < y \leq b$, $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \epsilon$. Then $f(z + \delta) - f(z) \leq \epsilon$. Hence,

$$f(z) \leq f^+(z) = \inf_{y \in (z,b)} f(y) \leq f(z + \delta) \geq f(z) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we get $f^+(z) = f(z)$.

Now suppose $f^-(x) = f(x)$ for all $x \in (a, b]$ and $f^+(x) = f(x)$ for all $x \in [a, b)$. We want to show that f is uniformly continuous. If $f(a) = f(b)$, then $f(x) = f(a)$ for all $x \in [a, b]$, since f is monotone. so uniform continuity trivially holds. Now assume $f(a) < f(b)$.

Pick $0 < \epsilon < 1$. Let $n \in \mathbb{N}$ such that $n \geq 2(f(b) - f(a))/\epsilon$. For $0 \leq i \leq n$, define $y_i := f(a) + (f(b) - f(a))(i/n)$. Then $Q := (y_0, \ldots, y_n) \in \mathcal{P}(f(a), f(b))$. Let $x_0 := a$ and $x_n := b$. For all $i \in [n - 1]$, using Theorem 5, we can find x_i such that $f(x_i) = y_i$. Hence, $P := (x_0, \ldots, x_n) \in \mathcal{P}(a, b)$.

Let $\delta := \min_{i=1}^n (x_i - x_{i-1})$. Suppose $a \leq y < z \leq b$ such that $z - y \leq \delta$. If $x_{i-1} \leq y < z \leq x_i$, then $f(z) - f(y) \leq f(x_i) - f(x_{i-1}) = y_i - y_{i-1} = (f(b) - f(a))/n \leq \epsilon/2$. Now suppose $y < x_i < z$. Then $x_{i-1} \leq y < x_i < z \leq x_{i+1}$, so $f(z) - f(y) \leq f(x_{i+1}) - f(x_{i-1}) = y_{i+1} - y_{i-1} = 2(f(b) - f(a))/n \leq \epsilon$. Hence, f is uniformly continuous. □

2 Defining Weighted Darboux Integrals

Definition 5 (Weighted Darboux Integral)**.** *Let $a, b \in \mathbb{R}$ such that $a \leq b$. Let $f, W : [a, b] \to \mathbb{R}$ be functions where W is monotonic. Let $P := (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \ldots < x_n = b$. Then the jump, lower, and upper* weighted Darboux sums *of f on P with weight W are denoted by $J_{f,W}(P)$, $L_{f,W}(P)$, and $U_{f,W}(P)$, respectively. If $a = b$, then $P = (a)$ and $J_{f,W}(P) = L_{f,W}(P) = U_{f,W}(P) = 0$. Otherwise,*

$$
\begin{aligned}
J_{f,W}(P) &:= f(x_0)(W^+(x_0) - W(x_0)) + \sum_{i=1}^{n-1} f(x_i)(W^+(x_i) - W^-(x_i)) \\
&\quad + f(x_n)(W(x_n) - W^-(x_n)), \\
L_{f,W}(P) &:= J_{f,W}(P) + \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1})) \left(\inf_{x \in (x_{i-1}, x_i)} f(x) \right), \\
U_{f,W}(P) &:= J_{f,W}(P) + \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1})) \left(\sup_{x \in (x_{i-1}, x_i)} f(x) \right).
\end{aligned}
$$

Let $\mathcal{P}(a, b) := \{(x_0, \ldots, x_n) : n \in \mathbb{N}, a = x_0 < \ldots < x_n = b\}$. $\mathcal{P}(a, b)$ is called the set of partitions *of $[a, b]$. The lower and upper* weighted Darboux integrals *of f with weight W are*

$$
L_{f,W}(a, b) := \sup_{P \in \mathcal{P}(a,b)} L_{f,W}(P) \qquad U_{f,W}(a, b) := \inf_{P \in \mathcal{P}(a,b)} U_{f,W}(P)
$$

*If $L_{f,W}(a, b) = U_{f,W}(a, b)$, then f is said to be W-*Darboux integrable *on $[a, b]$, and we denote $L_{f,W}(a, b)$ and $U_{f,W}(a, b)$ by $\int_a^b f(x)dW(x)$.*

When W is the identity function, we write L_f and U_f instead of $L_{f,W}$ and $U_{f,W}$, respectively. If $L_f(a, b) = U_f(a, b)$, we denote them by $\int_a^b f(x)dx$.

3 Properties of Darboux Sums

Definition 6 (concatenation)**.** *For any $P := (x_0, \ldots, x_m) \in \mathcal{P}(a, c)$ and $Q := (y_0, \ldots, y_n) \in \mathcal{P}(c, b)$, define $P \mid Q := (x_0, \ldots, x_m, y_1, \ldots, y_n) \in \mathcal{P}(a, b)$ to be the* concatenation *of P and Q.*

Lemma 10. *Let $f, W : [a, b] \to \mathbb{R}$ and W be monotonic. Let $P \in \mathcal{P}(a, c)$ and $Q \in \mathcal{P}(c, b)$. Then $L_{f,W}(P \mid Q) = L_{f,W}(P) + L_{f,W}(Q)$ and $U_{f,W}(P \mid Q) = U_{f,W}(P) + U_{f,W}(Q)$.*

Proof. Let $P = (x_0, \ldots, x_n)$ and $Q = (y_0, \ldots, y_m)$. For $i \in [m + n] \setminus [n]$ define $x_i := y_{i-n}$.

Then $P \mid Q = (x_0, \ldots, x_{m+n})$.

$$\begin{aligned}
J_{f,W}(P) + J_{f,W}(Q) &= f(x_0)(W^+(x_0) - W(x_0)) + \sum_{i=1}^{n-1} f(x_i)(W^+(x_i) - W^-(x_i)) \\
&\quad + f(x_n)(W(x_n) - W^-(x_n)) + f(y_0)(W^+(y_0) - W(y_0)) \\
&\quad + \sum_{i=1}^{m-1} f(y_i)(W^+(y_i) - W^-(y_i)) + f(y_m)(W^+(y_i) - W^-(y_i)) \\
&= f(x_0)(W^+(x_0) - W(x_0)) + \sum_{i=1}^{m+n-1} f(x_i)(W^+(x_i) - W^-(x_i)) \\
&\quad + f(x_{m+n-1})(W(x_{m+n}) - W^-(x_{m+n})) \\
&= J_{f,W}(P \mid Q)
\end{aligned}$$

It's easy to see that $L_{f,W}(P \mid Q) - J_{f,W}(P \mid Q) = (L_{f,W}(P) - J_{f,W}(P)) + (L_{f,W}(Q) - J_{f,W}(Q))$, so $L_{f,W}(P \mid Q) = L_{f,W}(P) + L_{f,W}(Q)$. Similarly, $U_{f,W}(P \mid Q) = U_{f,W}(P) + U_{f,W}(Q)$. □

Lemma 11. *Let $f, W : [a,b] \to \mathbb{R}$ and W be monotonic. Let*

$$\alpha := \inf_{x \in (a,b)} f(x), \qquad \beta := \sup_{x \in (a,b)} f(x).$$

For any $P \in \mathcal{P}(a,b)$, we get

$$\begin{aligned}
&f(a)(W^+(a) - W(a)) + \alpha(W^-(b) - W^+(a)) + f(b)(W(b) - W^-(b)) \\
&\leq L_{f,W}(P) \leq U_{f,W}(P) \\
&f(a)(W^+(a) - W(a)) + \beta(W^-(b) - W^+(a)) + f(b)(W(b) - W^-(b)).
\end{aligned}$$

Proof. (Trivial) □

Lemma 12. *Let $f, W : [a,b] \to \mathbb{R}$ and W be monotonic. Let $\alpha \leq f(x) \leq \beta$ for all $x \in [a,b]$. Then for any $P \in \mathcal{P}(a,b)$, we get $\alpha(W(b) - W(a)) \leq L_{f,W}(P) \leq U_{f,W}(P) \leq \beta(W(b) - W(a))$.*

Proof. A trivial corollary of Theorem 11. □

Definition 7 (Refinement)**.** *Let $P_1 := (x_0, \ldots, x_n) \in \mathcal{P}(a,b)$ and $P_2 := (y_0, \ldots, y_m) \in \mathcal{P}(a,b)$. Then P_2 is a* refinement *of P_1 if for every i, there exists j such that $x_i = y_j$.*

Lemma 13. *Let $P, Q \in \mathcal{P}(a,b)$ such that Q is a refinement of P. Let $f, W : [a,b] \to \mathbb{R}$ and W be monotonic. Then $L_{f,W}(P) \leq L_{f,W}(Q) \leq U_{f,W}(Q) \leq U_{f,W}(P)$.*

Proof. Let $P = (x_0, \ldots, x_n)$ and $Q = (y_0, \ldots, y_m)$. Since Q is a refinement of P, we get that $m \geq n$ and there is a sequence $(i_0, \ldots, i_n) \in \mathcal{P}(0, m)$ such that $x_j = y_{i_j}$.

For $j \in [n]$, define $P_j := (x_{j-1}, x_j)$ and $Q_j := (y_k)_{k=i_{j-1}}^{i_j}$. Then we get $L_{f,W}(P_j) \leq L_{f,W}(Q_j) \leq U_{f,W}(Q_j) \leq U_{f,W}(P_j)$ using Theorem 11. Sum these inequalities over $j \in [n]$ and use Theorem 10 to get $L_{f,W}(P) \leq L_{f,W}(Q) \leq U_{f,W}(Q) \leq U_{f,W}(P)$. □

Lemma 14. *Let $f, g, W : [a,b] \to \mathbb{R}$ and W be monotonic and let $P \in \mathcal{P}(a,b)$. Then $L_{f+g,W}(P) \geq L_{f,W}(P) + L_{g,W}(P)$ and $U_{f+g,W}(P) \leq U_{f,W}(P) + U_{g,W}(P)$.*

Proof. It is easy to see that $J_{f+g,W}(P) = J_{f,W}(P) + J_{g,W}(P)$. Using Theorem 1, we get $L_{f+g,W}(P) - J_{f+g,W}(P) \geq (L_{f,W}(P) - J_{f,W}(P)) + (L_{g,W}(P) - J_{g,W}(P))$, and $U_{f+g,W}(P) - J_{f+g,W}(P) \leq (U_{f,W}(P) - J_{f,W}(P)) + (U_{g,W}(P) - J_{g,W}(P))$. □

Lemma 15. *Let $f, W : [a,b] \to \mathbb{R}$, W be monotonic, $z \in \mathbb{R} - \{0\}$, and $P \in \mathcal{P}(a,b)$. Then*

$$L_{zf,W}(P) = \begin{cases} zL_{f,W}(P) & \text{if } z > 0 \\ zU_{f,W}(P) & \text{if } z < 0 \end{cases}, \qquad U_{zf,W}(P) = \begin{cases} zU_{f,W}(P) & \text{if } z > 0 \\ zL_{f,W}(P) & \text{if } z < 0 \end{cases}.$$

Proof. Follows from Theorem 2. □

4 Properties of Darboux Integrals

Lemma 16. *Let $f, W : [a,b] \to \mathbb{R}$ and W be monotonic. Then*

$$U_{f,W}(a,b) - L_{f,W}(a,b) = \inf_{P \in \mathcal{P}(a,b)} (U_{f,W}(P) - L_{f,W}(P)) \geq 0.$$

Proof. For any $P \in \mathcal{P}(a,b)$, we have $U_{f,W}(a,b) \leq U_{f,W}(P)$ and $L_{f,W}(a,b) \geq L_{f,W}(P)$. Hence, $U_{f,W}(a,b) - L_{f,W}(a,b) \leq U_{f,W}(P) - L_{f,W}(P) \geq 0$. Since this is true for all $P \in \mathcal{P}(a,b)$, we get Hence, $U_{f,W}(a,b) - L_{f,W}(a,b) \leq \inf_{P \in \mathcal{P}(a,b)}(U_{f,W}(P) - L_{f,W}(P)) \geq 0$.

Let $\alpha := \inf_{P \in \mathcal{P}(a,b)}(U_{f,W}(P) - L_{f,W}(P))$. Let $\beta := U_{f,W}(a,b) - L_{f,W}(a,b)$. We want to show that $\beta \geq \alpha$. Assume $\beta < \alpha$. Let $0 < \epsilon < (\alpha - \beta)/2$. Then $\exists P_U \in \mathcal{P}$ such that $U_{f,W}(P_U) < U_{f,W}(a,b) + \epsilon$ and $\exists P_L \in \mathcal{P}$ such that $L_{f,W}(P_L) > L_{f,W}(a,b) - \epsilon$. Hence, $U_{f,W}(P_U) - L_{f,W}(P_L) \leq \beta + 2\epsilon < \alpha$.

Let $P = (x_0, \ldots, x_n)$ be a merger of P_L and P_U, i.e., for all i, $x_i \in P_L \cup P_U$. Then P is a refinement of both P_L and P_U. By Theorem 13, $U_{f,W}(P) \leq U_{f,W}(P_U)$ and $L_{f,W}(P_L) \leq L_{f,W}(P)$, so $U_{f,W}(P) - L_{f,W}(P) < U_{f,W}(P_U) - L_{f,W}(P_L) < \alpha$, which is a contradiction. Hence, $\beta \geq \alpha$, which implies

$$U_{f,W}(a,b) - L_{f,W}(a,b) = \inf_{P \in \mathcal{P}(a,b)} (U_{f,W}(P) - L_{f,W}(P)).$$

□

Lemma 17. *Let $f, W : [a,b] \to \mathbb{R}$, W be monotonic, and $a \leq c \leq b$. Then $L_{f,W}(a,b) = L_{f,W}(a,c) + L_{f,W}(c,b)$ and $U_{f,W}(a,b) = U_{f,W}(a,c) + U_{f,W}(c,b)$.*

Proof. For any $\epsilon > 0$, $\exists P_1 \in \mathcal{P}(a,c)$ such that $L_{f,W}(P_1) > L_{f,W}(a,c) - \epsilon$ and $\exists P_2 \in \mathcal{P}(c,b)$ such that $L_{f,W}(P_2) > L_{f,W}(c,b) - \epsilon$. Using Theorem 10, we get $L_{f,W}(a,b) \geq L_{f,W}(P_1 \mid P_2) > L_{f,W}(a,c) + L_{f,W}(c,b) - 2\epsilon$. By picking ϵ arbitrarily small, we get $L_{f,W}(a,b) \geq L_{f,W}(a,c) + L_{f,W}(c,b)$.

For any $P \in \mathcal{P}(a,b)$, there exists a refinement $\widehat{P}$ of P such that $c \in \widehat{P}$. Then $\widehat{P} = P_1 \mid P_2$ for some $P_1 \in \mathcal{P}(a,c)$ and $P_2 \in \mathcal{P}(c,b)$. By Theorem 13, we get $L_{f,W}(P) \leq L_{f,W}(\widehat{P}) = L_{f,W}(P_1) + L_{f,W}(P_2) \leq L_{f,W}(a,c) + L_{f,W}(c,b)$. Since this is true for all $P \in \mathcal{P}(a,b)$, we get $L_{f,W}(a,b) \leq L_{f,W}(a,c) + L_{f,W}(c,b)$.

Hence, $L_{f,W}(a,b) = L_{f,W}(a,c) + L_{f,W}(c,b)$. We can similarly prove that $U_{f,W}(a,b) = U_{f,W}(a,c) + U_{f,W}(c,b)$. □

Lemma 18. *Let $f, W : [a, b] \to \mathbb{R}$ and W be monotonic. Let*

$$\alpha := \inf_{x \in (a,b)} f(x), \qquad \beta := \sup_{x \in (a,b)} f(x).$$

Then

$$\begin{aligned}
&f(a)(W^+(a) - W(a)) + \alpha(W^-(b) - W^+(a)) + f(b)(W(b) - W^-(b)) \\
&\le L_{f,W}(a, b) \le U_{f,W}(a, b) \\
&f(a)(W^+(a) - W(a)) + \beta(W^-(b) - W^+(a)) + f(b)(W(b) - W^-(b)).
\end{aligned}$$

Proof. Follows from Theorem 11. □

Lemma 19. *Let $f, W : [a, b] \to \mathbb{R}$ and W be monotonic. Let α and β be lower and upper bounds, respectively, on f. Then $(W(b) - W(a))\alpha \le L_{f,W}(a, b) \le U_{f,W}(a, b) \le (W(b) - W(a))\beta$.*

Proof. Follows from Theorem 12. □

Lemma 20. *Let $f, g, W : [a, b] \to \mathbb{R}$ and W be monotonic. Then $L_{f+g,W}(a, b) \ge L_{f,W}(a, b) + L_{g,W}(a, b)$ and $U_{f+g,W}(a, b) \le U_{f,W}(a, b) + U_{g,W}(a, b)$.*

Proof. Let $P_1, P_2 \in \mathcal{P}(a, b)$. Let $P \in \mathcal{P}(a, b)$ be the merger of P_1 and P_2, i.e., $x \in P$ iff $x \in P_1 \cup P_2$. Then P is a refinement of P_1 and P_2. Using Theorems 13 and 14, we get $L_{f+g,W}(a, b) \ge L_{f+g,W}(P) \ge L_{f,W}(P) + L_{g,W}(P) \ge L_{f,W}(P_1) + L_{g,W}(P_2)$. By taking suprema over P_1 and P_2, we get $L_{f+g,W}(a, b) \ge L_{f,W}(a, b) + L_{g,W}(a, b)$. We can similarly prove $U_{f+g,W}(a, b) \le U_{f,W}(a, b) + U_{g,W}(a, b)$. □

Lemma 21. *Let $f, W : [a, b] \to \mathbb{R}$, W be monotonic, and $z \in \mathbb{R} - \{0\}$. Then*

$$L_{zf,W}(a, b) = \begin{cases} zL_{f,W}(a, b) & \text{if } z > 0 \\ zU_{f,W}(a, b) & \text{if } z < 0 \end{cases}, \quad U_{zf,W}(a, b) = \begin{cases} zU_{f,W}(a, b) & \text{if } z > 0 \\ zL_{f,W}(a, b) & \text{if } z < 0 \end{cases}.$$

Proof. Take supremum over P in Theorem 15. □

5 Conditions for Integrability

Lemma 22. *Let $f, W : [a, b] \to \mathbb{R}$ where W is strictly monotone (i.e, $a \le x < y \le b$ implies $W(x) < W(y)$). If f has no upper-bound, then $U_{f,W}(a, b) = \infty$. If f has no lower-bound, then $L_{f,W}(a, b) = -\infty$.*

Proof sketch. Let $P = (x_0, \ldots, x_n) \in \mathcal{P}(a, b)$. Then

$$\sup_{x \in [a,b]} f(x) = \max\left(\max_{i=0}^{n} f(x_i), \max_{i=1}^{n} \sup_{x \in (x_{i-1}, x_i)} f(x)\right).$$

If f is unbounded, one of the above must be ∞, which would make the corresponding term in $U_{f,W}(P)$ infinity. Since this is true for all $P \in \mathcal{P}(a, b)$, we get $U_{f,W}(a, b) = \infty$.

We can similarly prove that $L_{f,W}(a, b) = -\infty$ if f has no lower bound. □

Lemma 23. *Let $f, W : [a, b] \to \mathbb{R}$ be monotone functions. Then $L_{f,W}(a, b) = U_{f,W}(a, b)$.*

Proof. By Theorem 19, we get $(W(b) - W(a))f(a) \leq L_{f,W}(a, b) \leq U_{f,W}(a, b) \leq (W(b) - W(a))f(b)$. Then $L_{f,W}(a, b) = U_{f,W}(a, b)$ if $f(a) = f(b)$ or $W(a) = W(b)$. Hence, assume $W(a) < W(b)$ and $f(a) < f(b)$.

Pick any $\epsilon > 0$. By Theorem 8, there exists $P = (x_0, \ldots, x_n) \in \mathcal{P}(a, b)$ such that $W^-(x_i) - W^+(x_{i-1}) \leq \epsilon(W^-(b) - W^+(a)) \leq \epsilon(W(b) - W(a))$ for all $i \in [n]$. Now,

$$\begin{aligned}
&U_{f,W}(P) - L_{f,W}(P) \\
&= \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1}))\left(\sup_{x\in(x_{i-1},x_i)} f(x) - \inf_{x\in(x_{i-1},x_i)} f(x)\right) \\
&\leq \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1}))(f(x_i) - f(x_{i-1})). \\
&\leq \epsilon(W(b) - W(a))\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})). \\
&= \epsilon(W(b) - W(a))(f(b) - f(a)).
\end{aligned}$$

By picking an arbitrarily small ϵ, using Theorem 16, we get

$$U_{f,W}(a, b) - L_{f,W}(a, b) = \inf_{P\in\mathcal{P}(a,b)}(U_{f,W}(P) - L_{f,W}(P)) = 0. \qquad \square$$

Lemma 24. *Let $f, W : [a, b] \to \mathbb{R}$, where W is monotone and f is uniformly continuous. Then $L_{f,W}(a, b) = U_{f,W}(a, b)$.*

Proof. Pick $\epsilon > 0$. Since f is uniformly continuous, $\exists\delta > 0$ such that for all $a \leq x < y \leq b$, $y - x \leq \delta$ implies $|f(x) - f(y)| \leq \epsilon$.

Use Theorem 8 to construct $P = (x_0, \ldots, x_n) \in \mathcal{P}(a, b)$ such that for all $i \in [n]$, we have $x_i - x_{i-1} \leq \delta$. Then for all $i \in [n]$ and $x, y \in (x_{i-1}, x_i)$, we have $|f(x) - f(y)| \leq \epsilon$. By Theorems 2 and 3, we get

$$\sup_{x\in(x_{i-1},x_i)} f(x) - \inf_{x\in(x_{i-1},x_i)} f(x) = \sup_{x,y\in(x_{i-1},x_i)} f(x) - f(y) \leq \epsilon.$$

Hence,

$$\begin{aligned}
&U_{f,W}(P) - L_{f,W}(P) \\
&= \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1}))\left(\sup_{x\in(x_{i-1},x_i)} f(x) - \inf_{x\in(x_{i-1},x_i)} f(x)\right) \\
&\leq \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1}))\epsilon \\
&\leq \epsilon(W^-(b) - W^+(a)).
\end{aligned}$$

By making ϵ arbitrarily small and using Theorem 16, we get

$$U_{f,W}(a, b) - L_{f,W}(a, b) = \inf_{P\in\mathcal{P}(a,b)}(U_{f,W}(P) - L_{f,W}(P)) = 0. \qquad \square$$

Lemma 25. *Let $R \subseteq \mathbb{R}$. Let $g : [a,b] \to R$ and $f : R \to \mathbb{R}$ such that f is γ-lipschitz. Let $W : [a,b] \to \mathbb{R}$ be monotonic. Let $h = f \cdot g$ be the composition of f and g. Then $U_{h,W}(a,b) - L_{h,W}(a,b) \leq \gamma(U_{g,W}(a,b) - L_{g,W}(a,b))$, and for any $P \in \mathcal{P}(a,b)$, we get $U_{h,W}(P) - L_{h,W}(P) \leq \gamma(U_{g,W}(P) - L_{g,W}(P))$.*

Proof. Let $P \in \mathcal{P}(a,b)$. Then

$$\begin{aligned}
&U_{h,W}(a,b) - L_{h,W}(a,b) \leq U_{h,W}(P) - L_{h,W}(P) \\
&= \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1}))\left(\sup_{x\in(x_{i-1},x_i)} h(x) - \inf_{x\in(x_{i-1},x_i)} h(x)\right) \\
&= \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1})) \sup_{x,y\in(x_{i-1},x_i)} |f(g(x)) - f(g(y))| \\
&\hspace{20em}\text{(using Theorems 2 and 3)} \\
&\leq \sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1})) \sup_{x,y\in(x_{i-1},x_i)} \gamma|g(x) - g(y)| \\
&= \gamma\sum_{i=1}^{n}(W^-(x_i) - W^+(x_{i-1}))\left(\sup_{x\in(x_{i-1},x_i)} g(x) - \inf_{x\in(x_{i-1},x_i)} g(x)\right) \\
&\hspace{20em}\text{(using Theorems 2 and 3)} \\
&= \gamma(U_{g,W}(P) - L_{g,W}(P)).
\end{aligned}$$

Hence,

$$U_{h,W}(a,b) - L_{h,W}(a,b) \leq \gamma \inf_{P\in\mathcal{P}(a,b)}(U_{g,W}(P) - L_{g,W}(P)).$$

Using Theorems 2 and 3, we get

$$U_{g,W}(a,b) - L_{g,W}(a,b) = \inf_P U_{g,W}(P) - \sup_P L_{g,W}(P) = \inf_P(U_{g,W}(P) - L_{g,W}(P)).$$

Hence, $U_{h,W}(a,b) - L_{h,W}(a,b) \leq \gamma(U_{g,W}(a,b) - L_{g,W}(a,b))$. □

Lemma 26. *Let $W : [a,b] \to \mathbb{R}$ be monotonic. Let $f : [a,b] \to [-c,c]$ and $g : [a,b] \to [-d,d]$, where $c \geq 0$ and $d \geq 0$. Let $h(x) := f(x)g(x)$ for all $x \in [a,b]$. Then $U_{h,W}(a,b) - L_{h,W}(a,b) = c(U_{f,W}(a,b) - L_{f,W}(a,b)) + d(U_{g,W}(a,b) - L_{g,W}(a,b))$.*

Proof. Let $P \in \mathcal{P}(a,b)$. Then

$$
\begin{aligned}
&U_{h,W}(a,b) - L_{h,W}(a,b) \le U_{h,W}(P) - L_{h,W}(P) \\
&= \sum_{i=1}^{n} (W^-(x_i) - W^+(x_{i-1})) \left(\sup_{x \in (x_{i-1},x_i)} h(x) - \inf_{x \in (x_{i-1},x_i)} h(x) \right) \\
&= \sum_{i=1}^{n} (W^-(x_i) - W^+(x_{i-1})) \sup_{x,y \in (x_{i-1},x_i)} (f(x)g(x) - f(y)g(y)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(using Theorems 2 and 3)} \\
&= \sum_{i=1}^{n} (W^-(x_i) - W^+(x_{i-1})) \sup_{x,y \in (x_{i-1},x_i)} (g(x)(f(x) - f(y)) + f(y)(g(x) - g(y))) \\
&\le \sum_{i=1}^{n} (W^-(x_i) - W^+(x_{i-1})) \left(\sup_{x,y \in (x_{i-1},x_i)} c|f(x) - f(y)| + \sup_{x,y \in (x_{i-1},x_i)} d|g(x) - g(y)| \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Theorem 1)} \\
&= c(U_{f,W}(P) - L_{f,W}(P)) + d(U_{g,W}(P) - L_{g,W}(P)).
\end{aligned}
$$

Using Theorems 2 and 3, we get

$$U_{f,W}(a,b) - L_{f,W}(a,b) = \inf_P U_{f,W}(P) - \sup_P L_{f,W}(P) = \inf_P (U_{f,W}(P) - L_{f,W}(P))$$

and

$$U_{g,W}(a,b) - L_{g,W}(a,b) = \inf_P U_{g,W}(P) - \sup_P L_{g,W}(P) = \inf_P (U_{g,W}(P) - L_{g,W}(P)).$$

Hence, we get $U_{h,W}(a,b) - L_{h,W}(a,b) = c(U_{f,W}(a,b) - L_{f,W}(a,b)) + d(U_{g,W}(a,b) - L_{g,W}(a,b))$. □

6 Integration By Parts

Lemma 27. *Let $f, g : [a,b] \to \mathbb{R}$ be monotonic functions. Let $P := (x_0, \ldots, x_n) \in \mathcal{P}(a,b)$. Let $K := \sum_{i=1}^{n} [(f(x_i) - f^-(x_i))(g(x_i) - g^-(x_i)) - (f^+(x_{i-1}) - f(x_{i-1}))(g^+(x_{i-1}) - g(x_{i-1}))]$. Then $L_{f,g}(P) + U_{g,f}(P) = U_{f,g}(P) + L_{g,f}(P) = f(b)g(b) - f(a)g(a) + K$.*

Proof sketch. We prove this for the special case of $P = (a,b) \in \mathcal{P}(a,b)$. The general result follows from the special case by applying Theorem 10.

$$
\begin{aligned}
&L_{f,g}(P) + U_{g,f}(P) \\
=&f(a)(g^+(a) - g(a)) + f^+(a)(g^-(b) - g^+(a)) + f(b)(g(b) - g^-(b)) \\
&+g(a)(f^+(a) - f(a)) + g^-(b)(f^-(b) - f^+(a)) + g(b)(f(b) - f^-(b)) \\
=&f(a)g^+(a) - f(a)g(a) - f^+(a)g^+(a) + f(b)g(b) - f(b)g^-(b) \\
&+f^+(a)g(a) - f(a)g(a) + f^-(b)g^-(b) + f(b)g(b) - f^-(b)g(b) \\
=&f(b)g(b) - f(a)g(a) - (f^+(a) - f(a))(g^+(a) - g(a)) + (f(b) - f^-(b))(g(b) - g^-(b)) \\
=&f(b)g(b) - f(a)g(a) + K.
\end{aligned}
$$

Note that K is symmetric in f and g. Exchange f and g in the above result to get that $U_{f,g}(P) + L_{g,f}(P) = f(b)g(b) - f(a)g(a) + K$. □

Lemma 28. *Let $f, g : [a, b] \to \mathbb{R}$ be monotonic functions. (Then $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ exist by Theorem 23.) If either f or g is uniformly continuous, then $\int_a^b f(x)dg(x) + \int_a^b g(x)df(x) = f(b)g(b) - f(a)g(a)$.*

Proof. Pick any $P \in \mathcal{P}(a, b)$. Since either f or g is uniformly continuous, K (as defined in Theorem 27) is 0 by Theorem 9. Let $\alpha := f(b)g(b) - f(a)g(a)$. Hence, $L_{f,g}(P) + U_{g,f}(P) = U_{f,g}(P) + L_{g,f}(P) = \alpha$.

Pick any $\epsilon > 0$. We can find P_1 and P_2 such that $L_{f,g}(a, b) - \epsilon \le L_{f,g}(P_1) \le L_{f,g}(a, b)$ and $U_{g,f}(a, b) \le U_{g,f}(P_2) \le U_{g,f}(a, b) + \epsilon$. Let $P := P_1 \cup P_2$. Then P is a refinement of P_1 and P_2, so $L_{f,g}(P) \le L_{f,g}(a, b) \le L_{f,g}(P) + \epsilon$ and $U_{g,f}(P) - \epsilon \le U_{g,f}(a, b) \le U_{g,f}(P)$. On adding the two equations, we get $\alpha - \epsilon \le L_{f,g}(a, b) + U_{g,f}(a, b) \le \alpha + \epsilon$. Since this holds true for all $\epsilon > 0$, we get $L_{f,g}(a, b) + U_{g,f}(a, b) = \alpha$.

Swap f and g in the result above to get $U_{f,g}(a, b) + L_{g,f}(a, b) = \alpha$. ☐

7 Integrals over Infinitesimal Intervals

Lemma 29. *Let $f : [a, b] \to [0, M]$. Let $W : [a, b] \to \mathbb{R}$ be monotonic. For $x \in [a, b]$, let $\ell(x) := L_{f,W}(a, x)$ and $u(x) := U_{f,W}(a, x)$. (Then ℓ and u are monotonic by Theorem 17 and non-negativity of f.) Then for any $c \in [a, b)$, $\ell^+(c) - \ell(c) = u^+(c) - u(c) = f(c)(W^+(c) - W(c))$, and for any $c \in (a, b]$, $\ell(c) - \ell^-(c) = u(c) - u^-(c) = f(c)(W(c) - W^-(c))$.*

Proof. Let $c \in [a, b)$. For any $x \in (c, b)$, using Theorem 18, we get

$$\begin{aligned}
&L_{f,W}(c, x) \\
&\ge f(c)(W^+(c) - W(c)) + \left(\inf_{z \in (c,x]} f(z)\right)(W(x) - W^+(c)) \\
&\ge f(c)(W^+(c) - W(c)).
\end{aligned}$$

Hence, $\inf_{x \in (c,b)} L_{f,W}(c, x) \ge f(c)(W^+(c) - W(c))$. Now pick any $\epsilon > 0$. Using Theorem 6, we can pick $y \in (c, b)$ such that $W(y) - W^+(c) \le \epsilon(W^-(b) - W^+(c))$. Using Theorem 18,

$$\begin{aligned}
&\inf_{x \in (c,b)} U_{f,W}(c, x) \le U_{f,W}(c, y) \\
&\le f(c)(W^+(c) - W(c)) + \left(\sup_{z \in (c,y]} f(z)\right)(W(y) - W^+(c)) \\
&\le f(c)(W^+(c) - W(c)) + M\epsilon(W^-(b) - W^+(c)).
\end{aligned}$$

Since this is true for all $\epsilon > 0$, we get $\inf_{x \in (c,b)} U_{f,W}(c, x) \le f(c)(W^+(c) - W(c))$. Hence, $\inf_{x \in (c,b)} L_{f,W}(c, x) = \inf_{x \in (c,b)} U_{f,W}(c, x) = f(c)(W^+(c) - W(c))$. So,

$$\ell^+(c) - \ell(c) = \inf_{x \in (c,b)} (\ell(x) - \ell(c)) = \inf_{x \in (c,b)} L_{f,W}(c, x) = f(c)(W^+(c) - W(c)).$$

$$u^+(c) - u(c) = \inf_{x \in (c,b)} (u(x) - u(c)) = \inf_{x \in (c,b)} U_{f,W}(c, x) = f(c)(W^+(c) - W(c)).$$

Let $c \in (a, b]$. For any $x \in (a, c)$, using Theorem 18, we get

$$\begin{aligned}
&L_{f,W}(x, c) \\
&\geq \left(\inf_{z \in [x,c)} f(z) \right) (W^-(c) - W(x)) + f(c)(W(c) - W^-(c)) \\
&\geq f(c)(W(c) - W^-(c)).
\end{aligned}$$

Hence, $\inf_{x \in (a,c)} L_{f,W}(x, c) \geq f(c)(W(c) - W^-(c))$. Now pick any $\epsilon > 0$. Using Theorem 6, we can pick $y \in (a, c)$ such that $W^-(c) - W(y) \leq \epsilon(W^-(c) - W^+(a))$. Using Theorem 18,

$$\begin{aligned}
&\inf_{x \in (a,c)} U_{f,W}(x, c) \leq U_{f,W}(y, c) \\
&\leq \left(\sup_{z \in [y,c)} f(z) \right) (W^-(c) - W(y)) + f(c)(W(c) - W^-(c)) \\
&\leq f(c)(W(c) - W^-(c)) + M\epsilon(W^-(c) - W^+(a)).
\end{aligned}$$

Since this is true for all $\epsilon > 0$, we get $\inf_{x \in (a,c)} U_{f,W}(x, c) \leq f(c)(W(c) - W^-(c))$. Hence, $\inf_{x \in (a,c)} L_{f,W}(x, c) = \inf_{x \in (a,c)} U_{f,W}(x, c) = f(c)(W(c) - W^-(c))$. So,

$$\ell(c) - \ell^-(c) = \inf_{x \in (a,c)} (\ell(c) - \ell(x)) = \inf_{x \in (a,c)} L_{f,W}(x, c) = f(c)(W(c) - W^-(c)).$$

$$u(c) - u^-(c) = \inf_{x \in (a,c)} (u(c) - u(x)) = \inf_{x \in (a,c)} U_{f,W}(x, c) = f(c)(W(c) - W^-(c)).$$

□

8 Double Integration

Lemma 30. *Let $V, W : [a, b] \to \mathbb{R}$ be monotonic functions and V be right continuous, i.e., $V^+(x) = V(x)$ for all $x \in [a, b)$. Let $f : [a, b] \to [0, M]$ for some $M \in \mathbb{R}_{\geq 0}$ such that $L_{f,V}(a, b) = U_{f,V}(a, b)$. Let $r : [a, b] \to \mathbb{R}_{\geq 0}$, where $r(x) := \int_a^x f(y)dV(y)$. (Then r is monotonic by f's non-negativity, so r is W-integrable over $[a, b]$.)*

Let $h(x) := f(x)(W(b) - W^-(x))$ for all $x \in (a, b]$, and let $h(a) := f(x)(W(b) - W(a))$. (Then h is V-integrable over $[a, b]$ by Theorem 26, since f is V-integrable and W is non-increasing.) Then $\int_a^b r(x)dW(x) = \int_a^b h(x)dV(x)$.

Proof. Suppose $W(b) = W(a)$. Then $W(x) = W(a) = W(b)$ for all $x \in [a, b]$. Hence, $\int_a^b r(x)dW(x) = 0$. Also, $h(x) = 0$ for all $x \in [a, b]$, so $\int_a^b h(x)dV(x) = 0$. Now assume $W(b) > W(a)$.

Pick any $P := (x_0, \ldots, x_n) \in \mathcal{P}(a, b)$. For $i \in [n]$, define $P_i := (x_0, \ldots, x_i) \in \mathcal{P}(a, x_i)$. Then for any $i \in [n]$, we get $L_{f,V}(P_i) \leq r(x_i) \leq U_{f,V}(P_i)$. Also, $r(a) = \int_a^a f(y)dV(y) = 0$. For $i \in [n]$, let $\alpha_i := \inf_{x \in (x_{i-1}, x_i)} f(x)$ and $\beta_i := \sup_{x \in (x_{i-1}, x_i)} f(x)$. Then

$$L_{f,V}(P_i) = \sum_{j=1}^{i} f(x_i)(V(x_j) - V^-(x_j)) + \sum_{j=1}^{n} \alpha_j(V^-(x_j) - V(x_{j-1})),$$

$$U_{f,V}(P_i) = \sum_{j=1}^{i} f(x_i)(V(x_j) - V^-(x_j)) + \sum_{j=1}^{n} \beta_j(V^-(x_j) - V(x_{j-1})).$$

Part 1: $L_{h,V}(a,b) \leq L_{r,W}(a,b)$

By Theorem 29, $r^+(x) = r(x)$ for all $x \geq 0$.

$$\begin{aligned}
L_{r,W}(a,b) &\geq L_{r,W}(P) \\
&= \sum_{i=1}^{n} r(x_i)(W^-(x_{i+1}) - W^-(x_i)) \\
&\qquad\qquad (W^-(x_{n+1}) := W(b) \text{ for notational convenience}) \\
&\geq \sum_{i=1}^{n} L_{f,V}(P_i)(W^-(x_{i+1}) - W^-(x_i)) \\
&= \sum_{j=1}^{n} (W(b) - W^-(x_j))(f(x_j)(V(x_j) - V^-(x_j)) + \alpha_j(V^-(x_j) - V(x_{j-1}))) \\
&:= K_L(P).
\end{aligned}$$

Now we will upper-bound $L_{h,V}(P)$.

$$\begin{aligned}
&\inf_{x\in(x_{i-1},x_i)} h(x) \\
&= \inf_{x\in(x_{i-1},x_i)} f(x)(W(b) - W^-(x)) \\
&\leq \inf_{x\in(x_{i-1},x_i)} \beta_i(W(b) - W^-(x)) \\
&= \beta_i \left(W(b) - \sup_{x\in(x_{i-1},x_i)} W^-(x) \right) \\
&= \beta_i \left(W(b) - (W^-)^-(x_i) \right) \\
&= \beta_i \left(W(b) - W^-(x_i) \right) \qquad \text{(by Theorem 4)}
\end{aligned}$$

Hence,

$$\begin{aligned}
L_{h,V}(P) &= \sum_{j=1}^{n} h(x_j)(V(x_j) - V^-(x_j)) + \sum_{j=1}^{n} \left(\inf_{x\in(x_{j-1},x_j)} h(x) \right) (V^-(x_j) - V(x_{j-1})) \\
&\leq \sum_{i=1}^{n} f(x_j)(W(b) - W^-(x_j))(V(x_j) - V^-(x_j)) \\
&\quad + \sum_{j=1}^{n} \beta_j(W(b) - W^-(x_j))(V^-(x_j) - V(x_{j-1})) \\
&= K_L(P) + \sum_{j=1}^{n} (\beta_j - \alpha_j)(W(b) - W^-(x_j))(V^-(x_j) - V(x_{j-1})) \\
&\leq K_L(P) + (W(b) - W(a)) \sum_{j=1}^{n} (\beta_j - \alpha_j)(V^-(x_j) - V(x_{j-1})) \\
&= K_L(P) + (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)). \\
&\leq L_{r,W}(a,b) + (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)).
\end{aligned}$$

Hence, for all $P \in \mathcal{P}(a,b)$, we have

$$L_{h,V}(P) \leq L_{r,W}(a,b) + (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)).$$

Pick any $\epsilon > 0$. Then $\exists Q_1 \in \mathcal{P}(a,b)$ such that $L_{h,V}(Q_1) > L_{h,V}(a,b) - \epsilon$. Since f is V-integrable, by Theorem 16, $\exists Q_2 \in \mathcal{P}(a,b)$ such that $U_{f,V}(Q_2) - L_{f,V}(Q_2) < \epsilon/(W(b) - W(a))$. Let $Q := Q_1 \cup Q_2$. Then Q is a refinement of Q_1 and Q_2, so by Theorem 13, we get $L_{h,V}(Q) \geq L_{h,V}(Q_1) > L_{h,V}(a,b) - \epsilon$ and $U_{f,V}(Q) - L_{f,V}(Q) \leq U_{f,V}(Q_2) - L_{f,V}(Q_2) < \epsilon/(W(b) - W(a))$. Hence, $L_{h,V}(a,b) - \epsilon \leq L_{h,V}(Q) \leq L_{r,W}(a,b) + (U_{f,V}(Q) - L_{f,V}(Q))(W(b) - W(a)) \leq L_{r,W}(a,b) + \epsilon$. Since this holds for all $\epsilon > 0$, we get $L_{h,V}(a,b) \leq L_{r,W}(a,b)$.

Part 1: $U_{h,V}(a,b) \geq U_{r,W}(a,b)$

By Theorem 29, $r^-(x) = r(x) - f(x)(V(x) - V^-(x))$ for all $x > 0$.

$$\begin{aligned}
U_{r,W}(a,b) &\leq U_{r,W}(P) \\
&= \sum_{i=1}^{n} r(x_i)(W^+(x_i) - W^+(x_{i-1})) \\
&\quad - \sum_{i=1}^{n} f(x_i)(V(x_i) - V^-(x_i))(W^-(x_i) - W^+(x_{i-1})) \\
&\qquad\qquad (W^+(x_n) := W(b) \text{ for notational convenience}) \\
&\leq \sum_{i=1}^{n} U_{f,V}(P_i)(W^+(x_i) - W^+(x_{i-1})) \\
&\quad - \sum_{i=1}^{n} f(x_i)(V(x_i) - V^-(x_i))(W^-(x_i) - W^+(x_{i-1})) \\
&= \sum_{j=1}^{n} f(x_j)(V(x_j) - V^-(x_j))(W(b) - W^-(x_j)) \\
&\quad - \sum_{j=1}^{n} \beta_j(V^-(x_j) - V(x_{j-1}))(W(b) - W^+(x_{j-1})) \\
&:= K_U(P).
\end{aligned}$$

$$\begin{aligned}
&\sup_{x \in (x_{i-1}, x_i)} h(x) \\
&= \sup_{x \in (x_{i-1}, x_i)} f(x)(W(b) - W^-(x)) \\
&\geq \sup_{x \in (x_{i-1}, x_i)} \alpha_i(W(b) - W^-(x)) \\
&= \alpha_i \left(W(b) - \inf_{x \in (x_{i-1}, x_i)} W^-(x) \right) \\
&= \alpha_i \left(W(b) - (W^-)^+(x_{i-1}) \right) \\
&= \alpha_i \left(W(b) - W^+(x_{i-1}) \right) && \text{(by Theorem 4)}
\end{aligned}$$

$$
\begin{aligned}
U_{h,V}(P) &\geq \sum_{j=1}^{n} h(x_j)(V(x_j) - V^-(x_j)) + \sum_{j=1}^{n} \left(\sup_{x \in (x_{j-1}, x_j)} h(x) \right) (V^-(x_j) - V(x_{j-1})) \\
&\geq \sum_{j=1}^{n} f(x_j)(W(b) - W^-(x_j))(V(x_j) - V^-(x_j)) \\
&\quad + \sum_{j=1}^{n} \alpha_j (W(b) - W^+(x_{j-1}))(V^-(x_j) - V(x_{j-1})) \\
&= K_U(P) - \sum_{j=1}^{n} (\beta_j - \alpha_j)(W(b) - W^+(x_{j-1}))(V^-(x_j) - V(x_{j-1})) \\
&\geq K_U(P) - (W(b) - W(a)) \sum_{j=1}^{n} (\beta_j - \alpha_j)(V^-(x_j) - V(x_{j-1})) \\
&= K_U(P) - (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)). \\
&\geq U_{r,W}(a,b) - (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)).
\end{aligned}
$$

Hence, for all $P \in \mathcal{P}(a,b)$, we have

$$
U_{h,V}(P) \geq U_{r,W}(a,b) - (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)).
$$

Pick any $\epsilon > 0$. Since f is V-integrable, using Theorems 13 and 16, we get that for some $Q \in \mathcal{P}(a,b)$, we have $U_{h,V}(Q) < U_{h,V}(a,b) + \epsilon$ and $U_{f,V}(Q) - L_{f,V}(Q) < \epsilon/(W(b) - W(a))$. Hence, $U_{h,V}(a,b) + \epsilon > U_{h,V}(Q) \geq U_{r,W}(a,b) - (W(b) - W(a))(U_{f,V}(P) - L_{f,V}(P)) \geq U_{r,W}(a,b) - \epsilon$. Hence, $U_{h,V}(a,b) \geq U_{r,W}(a,b) - 2\epsilon$. Since this holds for all $\epsilon > 0$, we get $U_{h,V}(a,b) \geq U_{r,W}(a,b)$.

Combining parts 1 and 2, and using the fact that r is W-integrable and h is V-integrable, we get that $\int_a^b r(x)dW(x) = \int_a^b h(y)dV(y)$. □

9 Rate Sandwich Lemma

Lemma 31 (Rate Sandwich Lemma)**.** *Let $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be functions where f is monotone. Then the following are equivalent:*

1. $f(y) \leq \dfrac{g(z) - g(y)}{z - y} \leq f(z)$ *for all* $0 \leq y < z$.
2. $\int_0^t f(x)dx = g(t) - g(0)$ *for all* $t \geq 0$.

Proof. Suppose $\int_0^t f(x)dx = g(t) - g(0)$ for all $t \geq 0$. Then for any $0 \leq y < z$, using Theorems 17, 19 and 23, we get

$$
g(z) - g(y) = \int_y^z f(x)dx \in [f(y)(z-y), f(z)(z-y)].
$$

Now suppose

$$
f(z) \geq \frac{g(z) - g(y)}{z - y} \geq f(y)
$$

for all $0 \leq y < z$. Let $t \in \mathbb{R}_{\geq 0}$. Let $P := (x_0, x_1, \ldots, x_n) \in \mathcal{P}(0,t)$. For any $i \in [n]$, we get

$$f(x_{i-1}) \leq \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \leq f(x_i).$$

Then the lower Darboux sum is

$$L_f(P) := \sum_{i=1}^{n} (x_i - x_{i-1}) f(x_{i-1}) \leq \sum_{i=1}^{n} (g(x_i) - g(x_{i-1})) = g(t) - g(0),$$

and the upper Darboux sum is

$$U_f(P) := \sum_{i=1}^{n} (x_i - x_{i-1}) f(x_i) \geq \sum_{i=1}^{n} (g(x_i) - g(x_{i-1})) = g(t) - g(0).$$

Hence, $L_f(P) \leq g(t) - g(0) \leq U_f(P)$. Hence, $L_f(0,t) \leq g(t) - g(0) \leq U_f(0,t)$. By Theorem 23, $U_f(0,t) = L_f(0,t)$, so we get $\int_0^t f(x)dx = g(t) - g(0)$. □