Chapter 1: Preliminaries

1 Sets

Definition 1 (Set basics)**.**

1. $A \subseteq B \;:\Longleftrightarrow\; (\forall x \in A, x \in B)$.
2. $A = B \;:\Longleftrightarrow\; (A \subseteq B \wedge B \subseteq A)$.
3. $A \subset B \;:\Longleftrightarrow\; (A \subseteq B \wedge B \nsubseteq A)$.
4. $A \cup B := \{x : x \in A \text{ or } x \in B\}$.
5. $A \cap B := \{x \in A : x \in B\}$.
6. $A \setminus B := \{x \in A : x \notin B\}$.
7. $\bigcup_{i \in I} A_i := \{x : \exists i \in I \text{ such that } x \in A_i\}$.
8. $\bigcap_{i \in I} A_i := \{x : \forall i \in I, x \in A_i\}$.
9. $A \times B := \{(x, y) : x \in A, y \in B\}$.
10. $\prod_{i=1}^{n} A_i := \{(x_1, x_2, \ldots, x_n) : x_i \in A_i \text{ for all } i\}$.

Theorem 1.

1. $A \subseteq B \iff A \cap B = A \iff A \cup B = B$.
2. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
3. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
4. $A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$.
5. $A \cap (B \cap C) = (A \cup B) \cap (A \cup C)$.

2 Relations and Functions

Definition 2 (Relation and function)**.** *A relation R between A and B is a subset of $A \times B$. A function $f : A \to B$ is a relation between A and B such that*

$$(a, b_1) \in f \text{ and } (a, b_2) \in f \implies b_1 = b_2.$$

$D(f) := A$ (called domain *of f), and $R(f) := B$ (called* range *of f).*

Lemma 2. *Let $f : A \to B$ and $g : A \to B$. Then $f = g \iff (\forall x \in A, f(x) = g(x))$.*

Definition 3 (Image and reverse image)**.** *Let $f : A \to B$ be a function.*

1. *For $X \subseteq A$, $f(X) := \{f(x) : x \in X\}$ is called the* image *of X under f. Equivalently, $y \in f(X) \iff (\exists x \in X, f(x) = y)$.*
2. *For $Y \subseteq B$, $f^{-1}(Y) = \{x : f(x) \in Y\}$ is called the* inverse image *of Y under f. Equivalently, $x \in f^{-1}(Y) \iff f(x) \in Y$.*

Lemma 3. *Let $f : A \to B$. Let $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$.*

1. $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$.
2. $f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$.
3. $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$.
4. $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

Definition 4 (Composition)**.** *For functions $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ is the defined as $(g \circ f)(x) = g(f(x))$.*

Definition 5 (Injection and surjection)**.** *Let $f : A \to B$.*

1. *f is injective (aka one-to-one) $:\iff \forall x_1 \in A, \forall x_2 \in A, (f(x_1) = f(x_2) \implies x_1 = x_2)$.*
2. *f is surjective (aka onto) $:\iff \forall y \in B, \exists x \in A, f(x) = y$.*

Lemma 4 (Composition)**.** *Let $f : A \to B$ and $g : B \to C$.*

1. *If f and g are injective, then $g \circ f$ is injective.*
2. *If $g \circ f$ is injective, then f is injective.*
3. *If f and g are surjective, then $g \circ f$ is surjective.*
4. *If $g \circ f$ is surjective, then g is surjective.*

Definition 6 (Identity)**.** *The identity function $\mathrm{id}_A : A \to A$ is given by $\mathrm{id}_A(x) = x$ for all $x \in A$.*

Definition 7 (Bijection)**.** *A function $f : A \to B$ is a bijection iff (the following are equivalent):*

1. *f is injective and surjective.*
2. *$\exists g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. (Then g is called the* inverse *of f, and is denoted by f^{-1}.)*

Proof sketch of equivalence. If f is injective and surjective, for each $y \in B$, there is a unique $x \in A$ such that $f(x) = y$. Define $g(y) = x$ and show condition 2. To show that condition 2 implies condition 1, use Lemma 4. □

Definition 8 (Restriction)**.** *Let $f : A \to B$ be a function. Let $X \subseteq A$. Then $f|X$ is a function from X to B such that $(f|X)(x) = f(x)$ for all $x \in X$.*

Lemma 5. *There is an injection from A to B iff there is a surjection from B to A.*

Proof. $\Longrightarrow$:
Let $f : A \to B$ be injective. Then for any $y \in B$, $f^{-1}(y)$ contains 0 or 1 elements. Let a be an arbitrary element of A. Define $g : B \to A$ as:

$$g(y) = \begin{cases} x & \text{if } f^{-1}(y) = \{x\} \\ a & \text{if } f^{-1}(y) = \emptyset \end{cases}.$$

Pick any $x \in A$. Let $y = f(x)$. Then $x \in f^{-1}(y)$. Hence, $g(y) = x$. Hence, g is surjective.

$\Longleftarrow$:
Let $g : B \to A$ be surjective. Then for any $x \in A$, we have $g^{-1}(x) \neq \emptyset$. By the axiom of choice, there is a function $f : A \to B$ such that $f(x) \in g^{-1}(x)$. For any distinct $x_1, x_2 \in A$, we have $g^{-1}(x_1) \cap g^{-1}(x_2) = \emptyset$. Hence, f is injective. □

3 Set Cardinality

Definition 9. *A set S is* countable *iff there exists a surjection $f : \mathbb{N} \to S$. A set S is* denumerable *iff it is countable and infinite.*

Lemma 6. *A set S is* countable *iff there exists a surjection $f : T \to S$, where T is countable.*

Lemma 7. *There is a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. (Hence, $\mathbb{N} \times \mathbb{N}$ is countable.)*

Proof sketch. The bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is defined as

Set $j = 1$
for $s \in \mathbb{N}$ **do**
    for i from 1 to $s - 1$ **do**
        Set $f(j) = (i, s - i)$
        j += 1
    end for
end for

□

Theorem 8. *Let I be a finite set and for each $i \in I$, let A_i be a countable set. Then $\prod_{i \in I} A_i$ is countable.*

Proof sketch. When $|I| = 2$, we can show a surjection from $\mathbb{N} \times \mathbb{N}$ to $A_1 \times A_2$. For larger $|I|$, use induction on $|I|$. $\square$

Theorem 9. *Let I be a countable set and for each $i \in I$, let A_i be a countable set. Then $\bigcup_{i \in I} A_i$ is countable.*

Proof sketch. Since I is countable, there exists a surjection $f : \mathbb{N} \to I$. Since A_i is countable, there exists a surjection $g_i : \mathbb{N} \to A_i$. Then $h(i, j) = g_i(j)$ is a surjection from $\mathbb{N} \times \mathbb{N}$ to $\bigcup_{i \in I} A_i$. $\square$

Theorem 10. *Let A be a countable set. Let B be the set of all finite subsets of A. Then B is countable.*

Proof sketch. Express B as union of k-sized sets and use Theorem 9. For each k, a subset is finite by Theorem 8. $\square$

Theorem 11 (Cantor's theorem)**.** *Let P be the power set of A. Then there is no surjection from A to P.*

Proof sketch. Assume there is a surjection $f : A \to P$. Let $D := \{a \in A : a \notin f(a)\}$. Since $D \in P$ and f is a surjection, we have $D = f(a_0)$ for some $a_0 \in A$. For each case $a_0 \in D$ and $a_0 \notin D$, show a contradiction. $\square$