Stochastic Processes

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Definition 1 (Stochastic Process). *Let $\mathcal{T} \subseteq \mathbb{R}$. For any $t \in \mathcal{T}$, let X_t (or $X(t)$) be a random variable with support D. Then $X := \{X_t : t \in \mathcal{T}\}$ is called a* stochastic process *on state-space D and time $\mathcal{T}$. Usually, $\mathcal{T}$ is either $\mathbb{Z}_{\geq 0}$ (discrete-time) or $\mathbb{R}_{\geq 0}$ (continuous-time).*

1 Discrete-Time Markov Chains

Definition 2 (Markov Chain). *Let $X := [X_0, X_1, \ldots]$ be a stochastic process on state-space D and time $\mathbb{Z}_{\geq 0}$. X is called a discrete-time markov chain if $\Pr(X_{t+1} = d \mid X_t, X_{t-1}, \ldots, X_0) = \Pr(X_{t+1} = d \mid X_t)$. If $\Pr(X_{t+1} = v \mid X_t = u) = \Pr(X_1 = v \mid X_0 = u)$ for all t, u, v, then X is called* time-homogeneous.

Definition 3 (Transition function). *Let X be a markov chain on state space D. Define $P^{(k)} : D \times D \mapsto [0, 1]$ as $P^{(k)}(i, j) = \Pr(X_k = j \mid X_0 = i)$. Then $P^{(k)}$ is called the k-step transition function of X. For $k = 1$, we simply write P instead of $P^{(1)}$. For a finite state space, we can represent P as a matrix.*

Lemma 1 (Chapman-Kolmogorov Equation). $P^{(m+n)}(i, j) = \sum_k P^{(m)}(i, k)P^{(n)}(k, j)$.

1.1 Classification of States, Recurrence, Limiting Probabilities

Definition 4. *Let $f_{i,j} := \Pr\left(\bigvee_{t\geq 1}(X_t = j) \,\middle|\, X_0 = i\right)$. Then $f_{i,j}$ is called the* eventual transition probability *from i to j. If $i = j$, then we write $f_{i,i}$ as f_i, and call it the* recurrence probability *of state i.*

Definition 5. *For a state i, let N_i be the random variable that counts the number of times we are in state i, i.e., $N_i := \sum_{t=0}^{\infty} \mathbf{1}(X_t = i)$. Then N_i is called the visit-count of i.*

Definition 6. *A state i of a markov chain is* recurrent *iff (the following are equivalent):*

- *the recurrence probability (f_i) of i is 1.*
- *i is visited infinitely often, i.e., $\Pr(N_i = \infty \mid X_0 = i) = 1$.*
- *i is visited infinitely often in expectation, i.e., $\mathrm{E}(N_i \mid X_0 = i) = \infty$.*

A non-recurrent state is called a transient *state.*

Lemma 2. $\Pr(N_i = k \mid X_0 = i) = f_i^{k-1}(1 - f_i)$.

Lemma 3. $\mathrm{E}(N_i \mid X_0 = i) = 1/(1 - f_i) = \sum_{t=0}^{\infty} P^{(t)}(i, i)$.

Definition 7. *State j is* accessible *from state i if $P^{(t)}(i,j) > 0$ for some t. States i and j* communicate *(denoted as $i \leftrightarrow j$) if i and j are both accessible from each other.*

Lemma 4. *Accessibility is reflexive and transitive. Communication is an equivalence relation. The equivalence classes of communicability are called* state classes. *A markov chain is* irreducible *if it has just one state class.*

Definition 8. *Let T_i be the time when a markov chain moves to state i, i.e., $T_i := \min_{t \geq 1}(X_t = i)$. When conditioned on $X_0 = i$, T_i is called the* recurrence time *of i. State i is called* positive recurrent *if $\mathrm{E}(T_i \mid X_0 = i)$ is finite, otherwise it is* null recurrent.

Lemma 5. *Recurrence and positive recurrence are class properties, i.e., they are same for all states in a class.*

Lemma 6. *In a finite-state markov chain, all recurrent states are positive recurrent, and there is at least one recurrent state.*

Definition 9 (Periodicity)**.** *For a state i, its period is defined as $\gcd(\{t : \Pr(T_i = t \mid X_0 = i) > 0\})$. A state is* aperiodic *if its period is 1.*

Lemma 7. *Periodicity is a class property.*

Definition 10 (Ergodicity)**.** *A state is ergodic if it is positive recurrent and aperiodic. A markov chain is ergodic if all its states are ergodic.*

Lemma 8. *In an irreducible ergodic markov chain, for every state j, $\lim_{t\to\infty} P^{(t)}(j,i) = \pi_i$ for a unique real number π_i. π_i is called the* limiting probability *of state i. Furthermore, π_i is the unique solution to this system of equations: $\pi_i = \sum_j \pi_j P(j,i)$ for all i ($\pi = P^T\pi$ in matrix form) and $\sum_i \pi_i = 1$.*

Lemma 9. *In an irreducible ergodic markov chain, $\mathrm{E}(T_i \mid X_0 = i) = 1/\pi_i$.*

Corollary 9.1. *A state i is null recurrent iff $\pi_i = 0$.*

Theorem 10. *If the transition function of markov chain X is doubly-stochastic (i.e., each row and each column sums to 1), then the limiting probability of each state is $1/n$, where n is the number of states.*

1.2 Time-Reversibility

Definition 11. *For an irreducible ergodic markov chain X with limiting probabilities π. Let Y be a markov chain whose transition function is $Q(i,j) = P(j,i)(\pi_j/\pi_i)$. Then Y is called the* time-reversed *markov chain of X. X is called* time-reversible *if $Q = P$.*

Theorem 11. *Let X be a time-reversible markov chain with limiting probabilities π. Then π is the unique solution to this system of equations: $x_j P(j,i) = x_i P(i,j)$ for all states i and j, and $\sum_i x_i = 1$.*

Theorem 12. *If the transition function of markov chain X is symmetric, then X is time-reversible.*

1.3 Simple Random Walk

Let X be a TH MC on state space $S = I \cap \mathbb{Z}$, where I is an interval of $\mathbb{R}$ and

$$\Pr(X_1 = j \mid X_0 = i) = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \\ 0 & \text{if } j \neq i \end{cases}.$$

Lemma 13 (Martingale property)**.** *Let* $\forall t \geq 1$*. If* $p = 1/2$*, then* $\mathrm{E}(X_t \mid X_{t-1}) = X_{t-1}$*. Otherwise, for* $r := (1-p)/p$*,* $\mathrm{E}(r^{X_t} \mid X_{t-1}) = r^{X_{t-1}}$*.*

Lemma 14. *Let* $I = [0, b]$*,* $T := \min_{t \geq 0}(X_t \in \{0, b\})$*, and* $r := (1-p)/p$*. Then* $\forall i \in S$*,*

$$p_i := \Pr(X_T = b \mid X_0 = i) = \begin{cases} \dfrac{i}{b} & \text{if } p = 1/2 \\ \dfrac{r^i - 1}{r^b - 1} & \text{if } p \neq 1/2 \end{cases}$$

$$\mu_i := \mathrm{E}(T \mid X_0 = i) = \begin{cases} i(b-i) & \text{if } p = 1/2 \\ \dfrac{1}{1-2p}\left(i - b\dfrac{r^i - 1}{r^b - 1}\right) & \text{if } p \neq 1/2 \end{cases}$$

Proof sketch. $p_0 = 0$, $p_b = 1$, and $p_i := pp_{i+1} + (1-p)p_{i-1}$ $\forall i \in [b-1]$. Rearrange to get $p(p_{i+1} - p_i) = (1-p)(p_i - p_{i-1})$ $\forall i \in [b-1]$. Let $d_i := p_i - p_{i-1}$ $\forall i \in [b]$. Then

$$\forall i \in [b], \quad \frac{d_i}{d_1} = \prod_{j=1}^{i-1} \frac{d_{j+1}}{d_j} = \prod_{j=1}^{i-1} r = r^{i-1},$$

$$\forall i \in S, \quad p_i = \sum_{j=1}^{i} d_j = \sum_{j=1}^{i} d_1 r^{j-1} = d_1 \times \begin{cases} i & \text{if } p = 1/2 \\ \frac{r^i - 1}{r-1} & \text{if } p \neq 1/2 \end{cases}.$$

Since $p_b = 1$, we get $d_1 = 1/b$ if $p = 1/2$ and $d_1 = (r-1)/(r^b - 1)$ otherwise.

$\mu_0 = \mu_b = 0$, and $\mu_i = 1 + p\mu_{i+1} + (1-p)\mu_{i-1}$ $\forall i \in [b-1]$. Rearrange to get $p(\mu_{i+1} - \mu_i) = (1-p)(\mu_i - \mu_{i-1}) - 1$ $\forall i \in [b-1]$. Let $\nu_i := \mu_i - \mu_{i-1}$ $\forall i \in [b]$. Then $\forall i \in [b-1]$, $\nu_{i+1} = r\nu_i - 1/p$.

Case 1: $p = 1/2$

$$\forall i \in [b], \quad \nu_i - \nu_1 = \sum_{j=1}^{i-1} (\nu_{j+1} - \nu_j) = -2(i-1).$$

$$\forall i \in S, \quad \mu_i = \sum_{j=1}^{i} \nu_j = \sum_{j=1}^{i} (\nu_1 - 2(j-1)) = i(\nu_1 - (i-1)).$$

Since $\mu_b = 0$, we get $\nu_1 = b - 1$, and so $\mu_i = i(b-i)$.

Case 2: $p \neq 1/2$

$$\nu_{i+1} = r\nu_i - \frac{1}{p} = r\nu_i - \frac{1}{p}\left(\frac{r}{r-1} - \frac{1}{r-1}\right) \implies \nu_{i+1} - \frac{1}{p(r-1)} = r\left(\nu_i - \frac{1}{p(r-1)}\right).$$

$p(r-1) = 1-2p$. Hence,

$$\forall i \in [b], \quad \frac{\nu_i - \frac{1}{1-2p}}{\nu_1 - \frac{1}{1-2p}} = \prod_{j=1}^{i-1} \frac{\nu_{j+1} - \frac{1}{1-2p}}{\nu_j - \frac{1}{1-2p}} = r^{i-1}.$$

$$\forall i \in S, \quad \mu_i = \sum_{j=1}^{i} \nu_i = \sum_{j=1}^{i} \left(\frac{1}{1-2p} + r^{j-1} \left(\nu_1 - \frac{1}{1-2p} \right) \right).$$
$$= \frac{i}{1-2p} + \left(\nu_1 - \frac{1}{1-2p} \right) \frac{r^i - 1}{r-1}.$$

Set $\mu_b = 0$ to get the answer. □

Lemma 15 (Catalan number). *The number of balanced parentheses strings of length $2n$ is $\binom{2n}{n}\frac{1}{n+1}$.*

Lemma 16. *For all $n \ge 1$,*

$$\binom{2n}{n} \frac{\sqrt{n}}{4^n} \in \left[\frac{2\sqrt{\pi}}{e^2}, \frac{e}{\sqrt{2\pi}} \right].$$

Proof. Use Stirling's approximation: $n!e^n/(n^n\sqrt{n}) \in [\sqrt{2\pi}, e]$. □

Lemma 17. *Let $S = \mathbb{Z}$. If $p \ne 1/2$, then every state is transient. Otherwise, every state is null recurrent.*

Proof. All states communicate, so all states belong to the same class. Recurrence and positive recurrence are class properties.

Using Definition 6, we get that state 0 is recurrent iff $\mathrm{E}(N_0 \mid X_0 = 0)$, where N_0 is the number of times we visit state 0.

$$\begin{aligned}
\mathrm{E}(N_0 \mid X_0 = 0) &= \sum_{t=0}^{\infty} P^{(t)}(0,0) && \text{(by Lemma 2)} \\
&= \sum_{j=0}^{\infty} 2\binom{2j}{j} p^j (1-p)^j && \text{(by Lemma 15)} \\
&\le \frac{\sqrt{2}e}{\pi} \sum_{j=0}^{\infty} \frac{(4p(1-p))^j}{\sqrt{j}}. && \text{(by Lemma 16)}
\end{aligned}$$

$4p(1-p) \le 1$ and $p = 1/2 \iff 4p(1-p) = 1$. Hence, the series is convergent iff $p \ne 1/2$. Hence, state 0 is recurrent iff $p = 1/2$.

Let $p = 1/2$. Then $T := \min_{t\ge 0}(X_t = 0)$ and $T' := \min_{t \ge 1}(X_t = 0)$. Then state 0 is null recurrent iff $\mathrm{E}(T' \mid X_0 = 0) = \infty$. Since state 0 is recurrent, $\Pr(T' = \infty \mid X_0 = 0) =$

0. Hence, $0 = \Pr(T' = \infty \mid X_0 = 0) = \Pr(T = \infty \mid X_0 = 1)$.

$$\begin{aligned}
&\mathrm{E}(T' \mid X_0 = 0) = 1 + \mathrm{E}(T \mid X_0 = 1) \\
&= 1 + \sum_{j=0}^{\infty} (2j+1) \Pr(T = 2j+1 \mid X_0 = 1) && \text{(since } \Pr(T = \infty \mid X_0 = 1) = 0\text{)} \\
&= 1 + \sum_{j=0}^{\infty} \frac{2j+1}{j+1} \binom{2j}{j} \frac{1}{2^{2j+1}} && \text{(by Lemma 15)} \\
&\geq 1 + \frac{\sqrt{\pi}}{e^2} \sum_{j=0}^{\infty} \frac{1}{\sqrt{j}} && \text{(by Lemma 16)} \\
&= \infty.
\end{aligned}$$

Hence, 0 is null recurrent. □

Lemma 18. *Let $I = [0, \infty)$. Let $T := \min_{t \geq 0}(X_t = 0)$. Then for $i > 0$,*

$$\mu_i := \mathrm{E}(T \mid X_0 = i) = \begin{cases} i/(1-2p) & \text{if } p < 1/2 \\ \infty & \text{if } p \geq 1/2 \end{cases}.$$

$$p_i := \Pr(T \neq \infty \mid X_0 = i) = \begin{cases} 1 & \text{if } p \leq 1/2 \\ r^i & \text{if } p > 1/2 \end{cases}.$$

Proof sketch. $\mu_i = i\mu_1$ and $\mu_1 = 1 + p\mu_2 = 1 + 2p\mu_1$. Suppose $\mu_1 \neq \infty$. If $p > 1/2$, then $\mu_1 = 1/(1-2p) < 0$, which is a contradiction. If $p = 1/2$, then $\mu_1 = 1 + \mu_1$, which is a contradiction. Hence, $\mu_1 = \infty$ when $p \geq 1/2$.

$p_i = p_1^i$ and $p_1 = (1-p) + pp_2$. Solving these equations gives us $p_1 \in \{1, r\}$. When $p \leq 1/2$, we get $r \geq 1$, but $p_1 \in [0, 1]$ (since p_1 is a probability). Hence, $p_i = 1$ when $p \leq 1/2$.

Let $p < 1/2$. Then $\mu_1 \in \{\infty, 1/(1-2p)\}$. We will show that $\mu_1 \neq \infty$. Since $p_i = 1$ for all $i \geq 0$, we have $\Pr(T = \infty \mid X_0 = i) = 0$. Hence,

$$\begin{aligned}
\mu_1 = \mathrm{E}(T \mid X_0 = 1) &= \sum_{j=0}^{\infty} (2j+1) \Pr(T = 2j+1 \mid X_0 = 1) \\
&= \sum_{j=0}^{\infty} \frac{2j+1}{j+1} \binom{2j}{j} p^j (1-p)^{j+1} && \text{(by Lemma 15)} \\
&\leq 2(1-p) \frac{e}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(4p(1-p))^j}{\sqrt{j}}. && \text{(by Lemma 16)}
\end{aligned}$$

This series is convergent, since $4p(1-p) < 1$ for $p < 1/2$. Hence, $\mu_1 \neq \infty$, so $\mu_1 = 1/(1-2p)$.

Let $p > 1/2$. Consider the random walk Y on state space $\mathbb{Z}$ where $\forall t \geq 0$, $\Pr(Y_{t+1} = j+1 \mid Y_t = j) = p$ and $\Pr(Y_{t+1} = j-1 \mid Y_t = j) = 1-p$. Let $T' := \min_{t \geq 0}(Y_t = 0)$. Then $\Pr(T \neq \infty \mid X_0 = 1) = 1 \iff \Pr(T' \neq \infty \mid Y_0 = 1) = 1$. $\Pr(T' \neq \infty \mid Y_0 = -1) = 1$, using the $p < 1/2$ case. Hence, we return to state 0 in Y with probability 1. Hence, 0 is a recurrent state in Y, which is a contradiction. Hence, $p_1 = \Pr(T \neq \infty \mid X_0 = 1) \neq 1$. Hence, $p_1 = r$. □

2 Counting Process

Definition 12 (Counting Process)**.** *Let N be a stochastic process on state space $\mathbb{Z}_{\geq 0}$ and time $\mathbb{R}_{\geq 0}$. Then N is called a counting process if $N(0) = 0$ and $N(t)$ is monotone in t, i.e., $t_1 < t_2 \implies N(t_1) \leq N(t_2)$.*

Definition 13 (Independent increments)**.** *A counting process N has* independent increments *iff for any two disjoint intervals $(u_1, v_1]$ and $(u_2, v_2]$ in $\mathbb{R}_{\geq 0}$, the random variables $N(v_1) - N(u_1)$ and $N(v_2) - N(u_2)$ are independent.*

Definition 14 (Stationary increments)**.** *A counting process N has* stationary increments *iff for any $u \leq v$, the random variables $N(v) - N(u)$ and $N(v - u)$ have the same distribution.*

Definition 15 (Arrival and interarrival times)**.** *For a counting process N, for $i \in \mathbb{Z}_{\geq 0}$, define the i^{th} arrival time $S_i := \min_{t \geq 0}(N(t) = i)$. For $i \in \mathbb{Z}_{\geq 1}$, define the i^{th} interarrival time $T_i := S_i - S_{i-1}$.*

Lemma 19. *For a counting process N with arrival times S, $N(t) \geq n \iff S_n \leq t$.*

Definition 16 (Stopping time)**.** *Let $X = [X_1, X_2, \ldots]$ be a sequence of random variables. The random variable N is called a stopping time for X if for all $n \geq 0$, (the following two definitions are equivalent):*

- *$N = n$ is independent of $X_{n+1}, X_{n+2}, \ldots$.*
- *$N \leq n$ is independent of $X_{n+1}, X_{n+2}, \ldots$.*

Theorem 20 (Wald's identity)**.** *Let $X = [X_1, X_2, \ldots]$ be a sequence of random variables where $\mathrm{E}(X_i) = \mu$ for all i. Let N be a stopping time for X. Then*

$$\mathrm{E}\left(\sum_{i=1}^{N} X_i\right) = \mu\,\mathrm{E}(N).$$

Proof sketch. For all i, $N \geq i$ is independent of X_i, and $\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i \,\mathbf{1}(N \geq i)$. □

3 Poisson Process

Definition 17 (Poisson process)**.** *A counting process N is a Poisson process with rate function $\lambda : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ if N has independent increments and $N(t_2) - N(t_1) \sim \mathrm{Poisson}(\mu)$, where $\mu := \int_{t_1}^{t_2} \lambda(t)dt$. N is called* homogeneous *if $\lambda(t) = \lambda(0)$ for all t, otherwise it is called* inhomogeneous*. For a homogeneous process, we denote $\lambda(0)$ by λ.*

Lemma 21. *A Poisson process N is homogeneous iff it has stationary increments.*

Theorem 22 (Alternative definition of Poisson process)**.** *A counting process N is a Poisson process with continuous rate function λ iff N has independent and stationary increments and $\Pr(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$ and $\Pr(N(t+h) - N(t) \geq 2) = o(h)$.*

Proof sketch for homogeneous. Let $g(u,t) := \mathrm{MGF}_u(N(t)) = \mathrm{E}(e^{uN(t)})$. Show $g(u,t) = 1+\lambda t(e^u-1)+o(t)$ straightforwardly. Use calculus to show that $g(u,t) = \exp(e^{\lambda t}(e^u-1))$ (find derivative w.r.t t by computing $\lim_{h\to 0}(g(u,t+h)-g(u,t))/h$; this gets rid of $o(h)$). Conclude that $N(t) \sim \text{Poisson}(\lambda t)$ since $g(u,t)$ is MGF of Poisson(λt). □

Lemma 23. *For a homogeneous Poisson process N,*

$$\Pr(N(s) = a \mid N(s+t) = a+b) = \binom{a+b}{a}\left(\frac{s}{s+t}\right)^a\left(\frac{t}{s+t}\right)^b.$$

Theorem 24. *Let N be a counting process. Then N is a homogeneous Poisson process with rate λ iff all interarrival times are independent and distributed* Expo(λ).

Theorem 25 (Decomposition theorem 1)**.** *Let K be a finite set, and let $\{N_i : i \in K\}$ be independent Poisson processes, where N_i has rate function λ_i. Let $N := \sum_{i\in K} N_i$. Then N is a Poisson process with rate function $\sum_{i\in K}\lambda_i$.*

Theorem 26 (Decomposition theorem 2)**.** *Let N be a Poisson process with rate function λ. Let K be a finite set (called set of labels). Suppose the j^{th} event receives label $L_j \in K$, where $\Pr(L_j = i) = p_i(S_j)$ for some function $p_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, and $\{N, L_1, L_2, \dots\}$ are independent. For $i \in K$, let $N_i(t)$ be the number of events having label i, i.e, $N_i(t) = \sum_{j=1}^{N(t)} \mathbf{1}(L_j = i)$. Then N_i is a Poisson process with rate function $p_i\lambda$. Furthermore, all N_i are independent and if all p_i are constant, then $N_i(t) \mid N(t) \sim \mathrm{Binom}(N(t), p_i)$.*

Lemma 27. *Let $N^{(1)}$ and $N^{(2)}$ be independent homogeneous Poisson processes with rates λ_1 and λ_2. Then*

$$\Pr(S_n^{(1)} < S_m^{(2)}) = \sum_{i=n}^{n+m-1}\binom{n+m-1}{i}\frac{\lambda_1^i\lambda_2^{n+m-1-i}}{(\lambda_1+\lambda_2)^{n+m-1}}.$$

Proof sketch. Model as a continuous markov chain with state space (n_1, n_2), where n_i is the number of events of $N^{(i)}$ that have occurred. □

Theorem 28 (arrival times distributed as order statistics)**.** *Let $X = [X_1, X_2, \dots, X_n]$ be IID uniform variables over $[0,t]$. Let $U = \mathrm{sorted}(X)$. Let N be a homogeneous Poisson process. Let S_i be the i^{th} arrival time of N. Then conditioned on $N(t) = n$, the distribution of $[S_1, \dots, S_n]$ and U are identical.*

Lemma 29 (Excess and Residual)**.** *Let N be a Poisson process with rate function λ. Let S_i be the i^{th} arrival time. Let $Y(t) := S_{N(t)+1} - t$ and $R(t) := t - S_{N(t)}$. Then $Y(t) > s \iff N(t+s) - N(t) = 0$ and $R(t) > r \iff N(t) - N(t-r) = 0$. If N is homogeneous, we get $Y(t) \sim \mathrm{Expo}(\lambda)$ and $R(t) \sim \mathrm{Expo}(\lambda)$.*

4 Continuous-Time Markov Chain

Definition 18 (CTMC)**.** *Let $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$ be a stochastic process on discrete state-space D. X is called a continuous-time markov chain (CTMC) if $\Pr(X(t+s) = d \mid \{X(u) : 0 \leq u \leq s\}) = \Pr(X(t+s) = d \mid X(s))$ for all $s, t \in \mathbb{R}_{\geq 0}$. If $\Pr(X(t+s) = v \mid X(s) = u) = \Pr(X(t) = v \mid X(0) = u)$ for all u, v, s, t, then X is called* time-homogeneous *(TH) or* stationary.

Theorem 30 (Equiv defn of TH CTMC). *Let $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$ be a stochastic process on discrete state-space D. Let $Y(t) := \{X(u) : 0 \leq u < t\}$. Let $T_i^{(s)} := \min_{t \geq 0}(X(t+s) \neq i)$. Let $P_{i,j}^{(s)} := \Pr(X(s+T_i^{(s)}) = j \mid X(s) = i, Y(s))$. X is TH CTMC iff $(T_i^{(s)} \mid X(s) = i, Y(s)) \sim \text{Expo}(\nu_i)$, where ν_i is a constant that doesn't depend on s or $Y(s)$, and $P_{i,j}^{(s)}$ is a constant that doesn't depend on s or $Y(s)$.*

Since $T_i^{(s)}$ and $P_{i,j}^{(s)}$ don't depend on s, we simply write T_i and $P_{i,j}$. T_i is called the transition time *out of state i, ν_i is called the* transition rate *out of state i, and $P_{i,j}$ is the* probability of transitioning *from state i to state j.*

Let $q_{i,j} := \nu_i P_{i,j}$. Then $\nu_i = \sum_j q_{i,j}$.

Theorem 31 (Chapman-Kolmogorov DiffEqs). *For a TH CTMC X, let $P_{i,j}(t) := \Pr(X(t) = j \mid X(0) = i)$. Then*

- *Backward DiffEqs:* $\dfrac{dP_{i,j}(t)}{dt} = \sum_{k \neq i} q_{i,k} P_{k,j}(t) - \nu_i P_{i,j}(t)$.
- *Forward DiffEqs:* $\dfrac{dP_{i,j}(t)}{dt} = \sum_{k \neq j} P_{i,k}(t) q_{k,j} - P_{i,j}(t)\nu_j$.

Let $r_{i,j} := \begin{cases} q_{i,j} & \text{if } i \neq j \\ -\nu_i & \text{if } i = j \end{cases}$. Let the state space be $[n]$. Let R be a matrix where $R[i,j] = r_{i,j}$. Then CBKE becomes $P'(t) = RP(t)$ and CFKE becomes $P'(t) = P(t)R$.

Lemma 32. *CKBE $P'(t) = RP(t)$ solves to $P(t) = e^{Rt}$, where $e^A := \sum_{i=0}^{\infty} A^i/i!$ for any square matrix A. Suppose R has n eigenpairs $\{(\lambda_1, v_i) : i \in [n]\}$. Let P be a square matrix whose i^{th} column is v_i, and D be a diagonal matrix whose i^{th} diagonal entry is λ_i. Then $R = PDP^{-1}$, $e^{Rt} = Pe^{Dt}P^{-1}$, and $e^{Dt} = \text{diag}([e^{\lambda_1 t}, \ldots, e^{\lambda_n t}])$.*

Lemma 33. *Let X be a TH CTMC.*

$$\lim_{h \to 0} \frac{1 - P_{i,i}(h)}{h} = \nu_i \quad \forall i \qquad\qquad \lim_{h \to 0} \frac{P_{i,j}(h)}{h} = q_{i,j} \quad \forall i \neq j$$

Lemma 34 (Limiting probability). *In an irreducible positive-recurrent TH CTMC X, for every state j, $\lim_{t \to \infty} P_{j,i}(t) = P_i$ for a unique real number P_i. P_i is called the* limiting probability *of state i. Furthermore, P_i is the unique solution to $\sum_i P_i = 1$ and CK forward equations, i.e., $P_i \nu_i = \sum_{j \neq i} P_j q_{j,i}$.*

Lemma 35 (Limiting probability of embedded chain). *Let X be an irreducible positive-recurrent TH CTMC. Let Y be the sequence of states visited by X. Then Y is a discrete MC. Let P and π be the limiting probabilities of X and Y, respectively. Then $P_i = (\pi_i/\nu_i)/(\sum_j \pi_j/\nu_j)$ and $\pi_i = P_i\nu_i/(\sum_j P_j\nu_j)$.*

Definition 19. *A CTMC is time-reversible iff the corresponding embedded discrete-time MC is time-reversible.*

Lemma 36 (2-state). *For a CTMC on states $\{0, 1\}$, where $q_{0,1} = \lambda$ and $q_{1,0} = \mu$, we get*

$$P(t) = \frac{1}{\lambda + \mu}\left(\begin{bmatrix} \mu & \lambda \\ \mu & \lambda \end{bmatrix} + e^{-(\mu+\lambda)t}\begin{bmatrix} \lambda & -\lambda \\ -\mu & \mu \end{bmatrix}\right).$$

4.1 Birth and Death Process

Definition 20. *A birth-and-death (B&D) process is a TH CTMC X on state space $\mathbb{Z}_{\geq 0}$ where $q_{i,j} = 0$ if $j \notin \{i-1, i+1\}$. Let $\lambda_i := q_{i,i+1}$ for $i \geq 0$, $\mu_i := q_{i,i-1}$ for $i \geq 1$, $\mu_0 := 0$.*

$X(t)$ is called the population *at time t, λ_i is called the* birth rate *at population i, and μ_i is called the* death rate *at population i.*

Lemma 37. *Let X be a B&D process where $X(0) = n$. Let T_n be the time to reach state $n+1$, i.e., $T_n := \min_{t \geq 0}(X(t) = n+1)$. Then*

$$\mathrm{E}(T_n) = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n}\mathrm{E}(T_{n-1}) = \frac{1}{\lambda_n}\sum_{i=0}^{n}\prod_{j=1}^{i}\frac{\mu_{n-j+1}}{\lambda_{n-j}}.$$

$$\mathrm{Var}(T_n) = \frac{1}{\lambda_n(\lambda_n+\mu_n)^2} + \frac{\mu_n}{\lambda_n}\mathrm{Var}(T_{i-1}) + \frac{\mu_n}{\lambda_n+\mu_n}(\mathrm{E}(T_{n-1}) + \mathrm{E}(T_n))^2$$

Proof sketch. Let $I_i = \mathbf{1}(\text{next transition goes to state } i+1)$. Let X_i be the transition time out of state i. Then $I_i \sim \mathrm{Bernouilli}(\lambda_i/(\mu_i+\lambda_i))$, $X_i \sim \mathrm{Expo}(\lambda_i+\mu_i)$, and

$$\mathrm{E}(T_i \mid I_i) = \mathrm{E}(X_i) + (1-I_i)(\mathrm{E}(T_{i-1}) + \mathrm{E}(T_i))),$$
$$\mathrm{Var}(T_i \mid I_i) = \mathrm{Var}(X_i) + (1-I_i)(\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)).$$

□

CKBE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_i P_{i-1,j}(t) + \lambda_i P_{i+1,j}(t) - (\lambda_i+\mu_i)P_{i,j}(t).$$

CKFE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_{j+1} P_{i,j+1}(t) + \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j+\mu_j)P_{i,j}(t).$$

Theorem 38 (Limiting Probabilities)**.** *Let X be an irreducible B&D process on state space $D \subseteq \mathbb{Z}_{\geq 0}$ where $0 \in D$. For $n \in D$, let $\alpha_n := \prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}$. If $\sum_{i\in D}\alpha_i$ is finite, then $P_i = \alpha_i P_0$, and $P_0 = 1/\sum_{i\in D}\alpha_i$.*

Proof sketch. Use Lemma 34 and add adjacent equations. □

5 Renewal Theory

Definition 21. *Let $[X_1, X_2, \ldots]$ be a sequence of IID non-negative randvars, called* interarrival times*, such that $\Pr(X_1 = 0) < 1$ and $\Pr(X_1 = \infty) = 0$. Let $S_n := \sum_{i=1}^{n} X_i$ (called* arrival times*). Let $N(t) := \max_n(S_n \leq t)$. Then N is called a* renewal process *(note that it is a counting process).*

We let F and f denote the CDF and PDF/PMF of X_1, respectively. We let $F^{(n)}$ and $f^{(n)}$ denote the CDF and PDF/PMF of S_n, respectively.

Let R_i be the reward *obtained at time X_i for all $i \geq 1$, where all R_i are independent. Let $R(t) := \sum_{i=1}^{N(t)} R_i$. Then R is called a* renewal reward process*.*

Lemma 39. *For all $t \geq 0$, $\Pr(N(t) = \infty) = 0$. $\Pr(\lim_{t\to\infty} N(t) = \infty) = 1$.*

Proof. Let $\mu := \mathrm{E}(X_1)$. $\mu > 0$ since $\Pr(X_n = 0) < 1$.

$$\Pr\left(\lim_{t\to\infty} \frac{S_n}{n} = \mu\right) = 1. \qquad \text{(strong law of large numbers)}$$

$$N(t) = \infty \iff (\forall n, S_n \le t) \implies \lim_{t\to\infty} \frac{S_n}{n} = 0.$$

$\Pr(N(\infty) = \infty) = 1$ since $\Pr(X_1 = \infty) = 0$. □

Definition 22. *For a renewal process N, let $m_N(t) := \mathrm{E}(N(t))$. Then m_N is called the* mean-value function *of N. (If N is clear from context, we will write m instead of m_N.)*

Lemma 40. $m(t) = \sum_{n=1}^{\infty} \Pr(S_n \le t) = \sum_{n=1}^{\infty} F^{(n)}(t)$.

Theorem 41. *m uniquely characterizes F.*

Lemma 42. *$m(t)$ is finite for all t.*

Theorem 43 (Renewal equation)**.** *When interarrival times are continuous randvars,*

$$m(t) = F(t) + \int_0^t m(t-x) f(x) dx.$$

Proof sketch. Let $N'(t) := \max_n \left(\sum_{i=2}^{n+1} X_i \le t\right)$. Then N and N' are identically distributed and

$$N(t) = \begin{cases} 1 + N'(t - X_1) & \text{if } X_1 \le t \\ 0 & \text{if } X_1 > t \end{cases}.$$

Finally, $m(t) = \mathrm{E}(\mathrm{E}(N(t) \mid X_1))$. □

Corollary 43.1. *Let N be a renewal process where interarrival times are distributed* $\mathrm{Uniform}(0, 1)$*. Then for $0 \le t \le 1$, $m(t) = e^t - 1$.*

Theorem 44 (Limit theorems)**.** *For a renewal process N with $\mu := \mathrm{E}(X_1)$,*

$$\Pr\left(\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}\right) = 1. \qquad \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Theorem 45 (Limit theorems for rewards)**.** *For a renewal process N with rewards $\{R_i : i \in \mathbb{Z}_{\ge 1}\}$, let $\alpha := \mathrm{E}(R_1)$ and $\mu := \mathrm{E}(X_1)$. Then*

$$\Pr\left(\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\alpha}{\mu}\right) = 1. \qquad \lim_{t\to\infty} \frac{\mathrm{E}(R(t))}{t} = \frac{\alpha}{\mu}.$$

Theorem 46 (Central limit theorem for renewals)**.** *For a renewal process N with $\mu := \mathrm{E}(X_1)$ and $\sigma^2 := \mathrm{Var}(X_1)$, the random variable*

$$\lim_{t\to\infty} \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}}$$

tends to the standard normal distribution.

Lemma 47 (Stopping time)**.** *Let $X = [X_1, X_2, \ldots]$ be the sequence of interarrival times for renewal process N. Then $N(t) + 1$ is a stopping time for X.*

Proof sketch. $N(t) + 1 \leq n \iff S_n > t$. □

Definition 23. *For a renewal process N with arrival times $S_1, S_2, \ldots$:*

- *Let $Y(t) := S_{N(t)+1} - t$. $Y(t)$ is called the* excess *at time t.*
- *Let $L(t) := t - S_{N(t)}$. $L(t)$ is called the* remaining life *at time t.*

Lemma 48. *Let N be a renewal process with interarrival times $X = [X_1, X_2, \ldots]$. Then $\mathrm{E}(S_{N(t)+1}) = t + \mathrm{E}(Y(t)) = \mathrm{E}(X_1)(m(t) + 1)$.*

Proof. $N(t) + 1$ is a stopping time for X. □