

# Random Walks

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## 1 One-dimensional random walk with left bounce and right absorb

There are  $n + 1$  nodes, numbered from 0 to  $n$ , where  $n \geq 1$ . Let  $X_t$  be a random variable denoting our position at time  $t$ . Let  $X_0 = 0$  (i.e., we start at node 0).

At node 0 we always move to node 1 in the next time step, i.e.,  $X_t = 0 \implies X_{t+1} = 1$ . Node  $n$  is absorbing, i.e.,  $X_t = n \implies X_{t+1} = n$ . At every other node  $i$ , we move to node  $i + 1$  with probability  $p$  and node  $i - 1$  with probability  $1 - p$ , where  $p \in (0, 1)$  is a constant. So for  $0 < i < n$ , we have

$$X_t = i \implies X_{t+1} = \begin{cases} i + 1 & \text{with probability } p \\ i - 1 & \text{with probability } 1 - p \end{cases}$$

Our aim is to find the expected number of moves to reach  $n$  from 0.

For  $0 \leq i < n$ , let  $s_i$  be the expected number of moves to reach  $i + 1$  from  $i$ . Then by linearity of expectation, the expected number of moves to reach  $n$  from 0 is  $\sum_{i=0}^{n-1} s_i$ .

Consider the sequence of nodes corresponding to a random walk that starts at  $i$  and ends at  $i + 1$ . Suppose this sequence contains  $t$  occurrences of node  $i$ . This means that we moved  $t - 1$  times from node  $i$  to node  $i - 1$  and we moved once from  $i$  to  $i + 1$ . The probability of observing such a sequence is  $(1 - p)^{t-1}p$ , which means that  $t$  is a geometric random variable. Therefore, the expected number of times we will move from  $i$  to  $i - 1$  is  $1/p - 1$ . Whenever we move from  $i$  to  $i - 1$ , we will have to random-walk our way back to  $i$  from  $i - 1$ , which will take  $s_{i-1}$  steps in expectation. Therefore,

$$s_i = 1 + \left(\frac{1}{p} - 1\right)(s_{i-1} + 1) = \frac{1 - p}{p}s_{i-1} + \frac{1}{p}$$

As a base case, we know that  $s_0 = 1$ . Therefore, we get

$$s_i = \begin{cases} 2i + 1 & \text{if } p = 1/2 \\ \frac{2(1-p)}{1-2p} \left(\frac{1-p}{p}\right)^i - \frac{1}{1-2p} & \text{if } p \neq 1/2 \end{cases}$$

When  $p = 1/2$ , the expected number of steps to reach  $n$  from 0 is

$$\sum_{i=0}^{n-1} s_i = \sum_{i=0}^{n-1} (2i + 1) = \sum_{i=0}^{n-1} ((i + 1)^2 - i^2) = n^2$$

When  $p \neq 1/2$ , the expected number of steps to reach  $n$  from 0 is

$$\sum_{i=0}^{n-1} s_i = \frac{2p(1-p)}{(1-2p)^2} \left( \left( \frac{1-p}{p} \right)^n - 1 \right) - \frac{n}{1-2p}$$

## 1.1 Alternative proof

Let  $Z_i$  be the expected number of steps needed to reach  $n$  from  $i$ . Then we have  $Z_n = 0$  and  $Z_0 = Z_1 + 1$ . For all other  $i$  from 1 to  $n-1$ , we either move to  $i+1$  with probability  $p$  and use  $Z_{i+1}$  steps to reach node  $n$ , or we move to  $i-1$  with probability  $1-p$  and use  $Z_{i-1}$  steps to reach node  $n$ . So,  $Z_i = 1 + pZ_{i+1} + (1-p)Z_{i-1}$ . Our aim is to find  $Z_0$ .

$$\begin{aligned} Z_i &= 1 + pZ_{i+1} + (1-p)Z_{i-1} \\ \implies Z_i - Z_{i+1} &= \frac{1}{p} + \frac{1-p}{p}(Z_{i-1} - Z_i) \end{aligned}$$

For  $p = 1/2$ , we get that for  $0 < j < n$ ,

$$\begin{aligned} Z_j - Z_{j+1} &= 2 + (Z_{j-1} - Z_j) \\ \implies \sum_{j=1}^i (Z_j - Z_{j+1}) &= \sum_{j=1}^i 2 + \sum_{j=1}^i (Z_{j-1} - Z_j) \\ \implies Z_1 - Z_{i+1} &= 2i + Z_0 - Z_i \\ \implies Z_i - Z_{i+1} &= 2i + 1 \\ \implies \sum_{i=1}^{n-1} (Z_i - Z_{i+1}) &= \sum_{i=1}^{n-1} (2i + 1) \\ \implies Z_1 - Z_n &= \sum_{i=1}^{n-1} ((i+1)^2 - i^2) = n^2 - 1 \\ \implies Z_0 &= n^2 \end{aligned}$$