

Bernstein-concentration

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Definition 1 (γ -bernstein-concentration). A random variable $X \in \mathbb{R}^n$ is γ -bernstein-concentrated iff

$$\Pr[a^T X - \mathbb{E}(a^T X) \geq t] \leq \exp\left(\frac{-1}{2\gamma} \frac{t^2}{\sum_{i=1}^n a_i^2 \text{Var}(X_i) + \frac{Mt}{3}}\right)$$

where $M = \max_{i=1}^n |a_i(X_i - \mathbb{E}(X_i))|$.

Theorem 1 (Bernstein's inequality). Let $X \in \mathbb{R}^n$ be a random variable where each coordinate of X is independent. Then X is 1-bernstein-concentrated.

1 Bernoulli variables

Theorem 2. Let $X \in \{0, 1\}^n$ be a random variable with $\mathbb{E}(X) = x$. If X is γ -bernstein-concentrated, then

$$\Pr[a^T X - a^T x \geq t] \leq \exp\left(\frac{-1}{2\gamma} \frac{t^2}{\|a\|_\infty (a^T x + t/3)}\right)$$

Proof. $\text{Var}(X_i) = x_i(1 - x_i)$.

$$\sum_{i=1}^n a_i^2 x_i(1 - x_i) \leq \|a\|_\infty \sum_{i=1}^n a_i x_i = \|a\|_\infty a^T x$$

$$M = \max_{i=1}^n |a_i| |X_i - \mathbb{E}(X_i)| \leq \|a\|_\infty \max_{i=1}^n |X_i - \mathbb{E}(X_i)| \leq \|a\|_\infty$$

□

Theorem 3. Let $X \in \{0, 1\}^n$ be a random variable with $\mathbb{E}(X) = x$. Let X be γ -bernstein-concentrated. Let $\alpha \leq a^T x \leq \beta$. Then

$$\Pr[a^T X \geq (1 + \delta)\beta] \leq \exp\left(\frac{-1}{2\gamma} \frac{\beta}{\|a\|_\infty} \frac{\delta^2}{1 + \delta/3}\right)$$

$$\Pr[a^T X \leq (1 - \delta)\alpha] \leq \exp\left(\frac{-1}{2\gamma} \frac{\alpha}{\|a\|_\infty} \frac{\delta^2}{1 + \delta/3}\right)$$

Proof. Let $c = (1 + \delta)\beta$.

$$\begin{aligned} \Pr[a^T X \geq c] &= \Pr[a^T X - a^T x \geq c - a^T x] \\ &\leq \exp\left(-\frac{1}{2\gamma} \frac{(c - a^T x)^2}{\|a\|_\infty (a^T x + \frac{1}{3}(c - a^T x))}\right) && \text{(by theorem 2)} \\ &\leq \exp\left(-\frac{1}{2\gamma} \frac{3}{\|a\|_\infty} \frac{(c - a^T x)^2}{c + 2a^T x}\right) \end{aligned}$$

Let $f(y) = (c - y)^2/(c + 2y)$, where $y \in [\alpha, \beta]$. $f(y) > 0$.

$$\ln(f(y)) = 2 \ln(c - y) - \ln(c + 2y)$$

$$\frac{f'(y)}{f(y)} = -\frac{2}{c - y} - \frac{2}{c + 2y} < 0$$

Since $f'(y) < 0$, $f(y)$ is minimized at $y = \beta$. $f(\beta) = (\delta^2\beta)/(3 + \delta)$. Therefore,

$$\Pr[a^T X \geq (1 + \delta)\beta] \leq \exp\left(\frac{-1}{2\gamma} \frac{\beta}{\|a\|_\infty} \frac{\delta^2}{1 + \delta/3}\right)$$

The other inequality can be proven similarly. □