Polyhedral Theory

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Prerequisites:

1. Convex combination, strict convex combination, convex hull, convex set.
2. Vector spaces, rank of a set of vectors.

1 Definitions

Definition 1 (Line and Ray). *For $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n - \{0\}$, a line is a set of the form $\{x + \lambda d : \lambda \in \mathbb{R}\}$, and a ray is a set of the form $\{x + \lambda d : \lambda \geq 0\}$.*

Definition 2 (Cone). *A set $S \in \mathbb{R}^n$ is called a cone if $\forall x \in S$ and $\forall \alpha \geq 0$, we have $\alpha x \in S$. A cone is called* pointed *if $\forall x \in S - \{0\}$, we have $-x \notin S$.*

Definition 3 (Bounded set). *A set S is bounded if $\exists \alpha \in \mathbb{R}$ such that $\|x\| \leq \alpha$ for all $x \in S$.*

Definition 4 (Vertex of a set). *Let $S \subseteq \mathbb{R}^n$ and $x \in S$. x is a vertex of S if $\exists c \in \mathbb{R}^n$ such that $\forall y \in S - \{x\}$, we have $c^T x > c^T y$.*

Definition 5 (Direction of a set). *d is a direction of $S \subseteq \mathbb{R}^n$ if $\forall x \in S$, $\forall \lambda \geq 0$, we have $x + \lambda d \in S$.*

Definition 6 (Extreme point). *$x \in S$ is an extreme point of $S \subseteq \mathbb{R}^n$ if (the following definitions are equivalent):*

1. *x is not a strict convex combination of at least 2 points in S.*
2. *x is not a strict convex combination of exactly 2 points in S.*
3. *$\forall y \in \mathbb{R}^n$, $x + y \notin S$ or $x - y \notin S$.*

Definition 7 (Extreme direction). *Let C be a pointed convex cone. Let $d \in C - \{0\}$. Then d is called an extreme direction of C iff d cannot be expressed as $x + y$, where $x \in C$ and $y \in C$ are non-collinear.*

Definition 8 (Projection). *Let $S \subseteq \mathbb{R}^n$. Let $R := \{(x_1, \ldots, x_k) : x \in S\}$ for any $k < n$. Then R is called the k-dimensional projection of S.*

Definition 9 (Minkowski sum). *Let S and T be two sets. Then their Minkowski sum $S + T$ is $\{s + t : s \in S, t \in T\}$.*

1.1 Polyhedra

Definition 10 (Polyhedron). *A polyhedron is an intersection of half-spaces. Formally, a polyhedron is represented using equalities and inequalities:* $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$.

Definition 11 (Polyhedral cone). *A polyhedral cone is (equiv defns):*

1. *a polyhedron that is also a cone.*
2. *a set of the form* $\{x \in \mathbb{R}^n : (a_i^T x \geq 0, \forall i \in I) \wedge (a_i^T x = 0, \forall i \in E)\}$.

Definition 12 (BFS). *Let Q be this system of inequations:* $(a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)$. $\widehat{x} \in \mathbb{R}^n$ *and let* $B := \{a_i : a_i^T \widehat{x} = b_i\}$. *Then $\widehat{x}$ is called a* basic feasible solution *(BFS) of Q if $\widehat{x}$ satisfies Q and* $\operatorname{rank}(B) = n$.

Definition 13 (BFD). *Let Q be this system of inequations:* $(a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)$. *Let Q' be this system of inequations:* $(a_i^T x \geq 0, \forall i \in I) \wedge (a_i^T x = 0, \forall i \in E)$. $\widehat{x} \in \mathbb{R}^n$ *and let* $B := \{a_i : a_i^T \widehat{x} = 0\}$. *Then $\widehat{x}$ is called a* basic feasible direction *(BFD) of Q if $\widehat{x}$ satisfies Q' and* $\operatorname{rank}(B) = n - 1$.

2 Results on Polyhedra

Theorem 1 (Fourier-Motzkin Elimination). *The projection of a polyhedron is also a polyhedron.*

Theorem 2 (Pointing a polyhedron). *Let* $P := \{x \in \mathbb{R}^n : (a_i^T x = b_i, \forall i \in E) \wedge (a_i^T x \geq b_i, \forall i \in I)\}$ *be a non-empty polyhedron. Let* $A := \{a_i : i \in I \cup E\}$. *Let* $B := \{a_i : i \in T\}$ *(where $T \subseteq I \cup E$) be a basis of A.*

Let $D := \{x \in \mathbb{R}^n : (a_i^T x = 0, \forall i \in I \cup E)\}$. *Let* $C := \{a_i : i \in H\}$ *(where H is a new set of indices) be a basis of D. Let* $P' := \{x \in P : (a_i^T x = 0, \forall i \in H)\}$.

Then the following hold:

1. $|C| = n - \operatorname{rank}(A)$ *and $B \cup C$ is a basis of $\mathbb{R}^n$.*
2. $P = \{\widehat{x} + d : \widehat{x} \in P', d \in D\}$.

Note that part 1 implies that P' is a full-rank polyhedron. Part 2 implies that any polyhedron P can be decomposed into a full-rank polyhedron P' and subspace D.

3 BFS and BFD

Theorem 3 (Equiv of extreme point, vertex, BFS). *Let* $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ *be a non-empty polyhedron. Then the following are equivalent for any* $\widehat{x} \in P$:

1. $\widehat{x}$ *is a vertex of P.*
2. $\widehat{x}$ *is an extreme point of P.*
3. $\widehat{x}$ *is a BFS of the system of constraints defining P.*

Theorem 4 (Equiv of BFD and extreme direction)**.** *Let* $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$*. Let* $C := \{x \in \mathbb{R}^n : (a_i^T x \geq 0, \forall i \in I) \wedge (a_i^T x = 0, \forall i \in E)\}$*.* d *is a BFD of* P*'s constraints iff it is an extreme direction of* C*.*

Theorem 5 (Condition for existence of BFS)**.** *Let* $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ *be a non-empty polyhedron. Let* $A = \{a_i : i \in I \cup E\}$*. Then the following are equivalent:*

1. P *has a BFS.*
2. P *doesn't contain a line.*
3. $\operatorname{rank}(A) = n$*.*

Theorem 6 (Representation theorem)**.** *Let* P *be a polyhedron. Let* $x^{(1)}, \ldots, x^{(p)}$ *be its BFSes and* $d^{(1)}, \ldots, d^{(q)}$ *be its BFDs, where* $p \geq 1$*. Then for any* $\widehat{x} \in P$*,* $\exists \lambda \in \mathbb{R}^p_{\geq 0}$*,* $\exists \mu \in \mathbb{R}^q_{\geq 0}$ *such that* $\sum_{i=1}^p \lambda_i = 1$ *and*

$$\widehat{x} = \sum_{i=1}^p \lambda_i x^{(i)} + \sum_{j=1}^q \mu_j d^{(j)}.$$

Proof sketch. First handle the special case of polytopes by inducting on the number of tight constraints. Reduce the special case of cones to the polytope case by scaling each non-zero point of the cone C to lie on a hyperplane H such that $C \cap H$ is bounded. For the general case, induct on the number of tight constraints, and use the result for cones. □

Theorem 7 (Converse of representation theorem)**.** *Let* $x^{(1)}, \ldots, x^{(p))}, d^{(1)}, \ldots, d^{(q)}$ *be points. Let* $P := \{\sum_{i=1}^p \lambda_i x^{(i)} + \sum_{j=1}^q \mu_j d^{(j)} : \lambda \in \mathbb{R}^p_{\geq 0}, \mu \in \mathbb{R}^q_{\geq 0}, \sum_{i=1}^p \lambda_i = 1\}$*. Then* P *is a polyhedron.*

Proof sketch. Simple application of Fourier-Motzkin Elimination. □

4 Duality and Farkas' Lemma

Content in this section is based on Prof. Karthik's notes: Lecture 3.

Theorem 8 (Fund thm of lin ineqs)**.** *Let* $a_1, \ldots, a_n, b \in \mathbb{R}^m$*. Let* $t := \operatorname{rank}(\{b, a_1, \ldots, a_n\})$*. Then exactly one of these holds:*

1. b *lies in the non-neg hull of* $\{a_i : i \in [n]\}$*.*
2. $\exists c \in \mathbb{R}^m$ *such that all of these hold:*
   (a) $c^T b < 0$*.*
   (b) $c^T a_i \geq 0$ *for all* i*.*
   (c) $\operatorname{rank}(\{a_i : a_i^T c = 0\}) = t - 1$*.*
   (d) *If* $\{b, a_1, \ldots, a_n\}$ *are rational, then* c *is rational.*

5 Dimension and Faces

Content in this section is based on Prof. Karthik's notes: Lectures 5 and 6.

Definition 14 (Implicit equality and redundant constraint). *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$. For $i \in I$, $a_i^T x \geq b_i$ is an implicit equality of P if $a_i^T x = b_i$ for all $x \in P$. For $i \in I \cup E$, the i^{th} constraint is redundant if removing it changes the polyhedron.*

Theorem 9 (dim from rank of constraints). *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ be a non-empty polyhedron without any implicit equalities. Let $A := \{a_i : i \in E\}$. Then $\dim(P) = n - \operatorname{rank}(A)$.*

Proof sketch. Use a max-cardinality affindep subset of P to construct a basis of A's nullspace. Use an interior point of P and a basis of A's nullspace to construct a max-cardinality affindep subset of P. □

Definition 15 (Face). *Let P be a non-empty polyhedron. F is a* face *of P if $F \neq \emptyset$ and $\exists c \in \mathbb{R}^n$, $\exists \beta \in \mathbb{R}$, such that $F = \{x \in P : c^T x = \beta\}$ and $(c^T x \geq \beta, \forall x \in P)$. (Note that c can be 0, in which case $F = P$.) A F is called a* proper face *of P if F is a face of P and $F \neq P$.*

Definition 16 (Tightening). *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ F is a tightening of P if $\exists T$ such that $E \subseteq T \subseteq I \cup E$ and $F = \{x \in \mathbb{R}^n : (a_i^T x = b_i, \forall i \in T) \wedge (a_i^T x \geq b_i, \forall i \in I - T)\}$. T is called the set of tight constraints for F.*

Lemma 10 (Tightening is face). *If F is a tightening of P, then F is a face of P.*

Proof sketch. Pick $c := \sum_{i \in T} a_i$ and $\beta := \sum_{i \in T} b_i$. Then $\forall x \in F$, $c^T x = \beta$, and $\forall x \in P - F$, $c^T x > \beta$. □

Theorem 11 (Face is tightening). *If F is a face of P, then F is a tightening of P.*

Proof sketch. Let $F := \{x \in P : c^T x = \beta\}$ be a face of P, where $c^T x \geq \beta$ for all $x \in P$. Consider the LP $\min_{x \in P} c^T x$. Then $\operatorname{opt}(\text{LP}) = \beta$ and $F = \operatorname{argopt}(\text{LP})$, since $F \neq \emptyset$. Let w^* be the optimal dual solution. Let $H := \{i \in I : w_i^* > 0\}$. Let $T := E \cup H$. Let $F' := \{x : (a_i^T x = b_i, \forall i \in T) \wedge (a_i^T x \geq b_i, \forall i \in I - T)\}$.

Let $\widehat{x} \in F'$. We can show that $(\widehat{x}, w^*)$ is a pair of solutions satisfying complementary slackness. Hence, $\widehat{x}$ is optimal, so $\widehat{x} \in \operatorname{argopt}(\text{LP}) = F$.

For all $\widehat{x} \in F$, $\widehat{x}$ is optimal for LP, and so $(\widehat{x}, w^*)$ satisfies complementary slackness (by strong duality of LPs). Hence, for $i \in H$, $a_i^T \widehat{x} = b_i$. Hence, $\widehat{x} \in F'$. Hence, $F = F'$. Since F' is a tightening, so is F. □

Definition 17 (Facet). *A facet of P is a maximal proper face of P, i.e., F is a facet of P if for every proper face F' of P, we have $F' \supseteq F \implies F' = F$.*

Theorem 12 (Facet is 1-tightening). *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ be a non-empty polyhedron without implicit equalities. If F is a facet of P, then $\exists k \in I$ such that $F = \{x \in P : a_k^T x = b_k\}$.*

Proof. T be the tight constraint indices for F. Pick any $k \in T$. Let $F' := \{x \in P : a_k^T x = b_k\}$. Then $F \subseteq F' \subseteq P$. Since k is not an implicit equality, $F' \neq P$. Since F is a maximal face, $F = F'$. □

Theorem 13 (1-tightening is facet). *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ be a non-empty polyhedron without implicit equalities. Let $R \subseteq I$ be the set of irredundant constraints. If $F = \{x \in P : a_k^T x = b_k\}$ for some $k \in R$, then F is a facet.*

Proof sketch. By Lemma 10, F is a face of P. Since k is not an implicit equality, $F \neq P$. Use irredundancy of k to find an interior point $\widehat{x}$ of F (pick an interior point $x^{(1)}$ of P, pick a point $x^{(2)}$ that violates only the k^{th} constraint, and interpolate between them).

Suppose F is not a facet. Then it is contained by another proper face $F' \neq F$. Let T' be the tight constraints of F'. Since $\widehat{x} \in F'$, $T' = E \cup \{k\}$, so $F = F'$. □

Theorem 14 (Facet has dimension $\dim(P) - 1$)**.** *Let F be a face of P. Then F is a facet of P iff* $\dim(F) = \dim(P) - 1$.

Proof sketch. Let E be the equality (both implicit and explicit) constraints. Let T be a set of tight constraints of F. If F is a facet, $|T - E| = 1$ by Theorem 12. Then use Theorem 9. Now let $\dim(F) = \dim(P) - 1$. Then $\operatorname{rank}(T) = \operatorname{rank}(E) + 1$. Let $k \in T - E$ be linindep from E. Then $\operatorname{span}(E \cup \{k\}) = \operatorname{span}(T)$, so $E \cup \{k\}$ is also a set of tight constraints for F. By Theorem 13, F is a facet. □

Theorem 15 (Minimal face from subsystem)**.** *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ be a non-empty polyhedron. Let $F \subseteq P$ and $F \neq \emptyset$. Then F is a minimal face of P iff $\exists K$ such that $E \subseteq K \subseteq I \cup E$ and $F = \{x : (a_i^T x = b_i, \forall i \in K)\}$.*

Proof. (TODO) □

Theorem 16 (Minimal face from rank)**.** *Let $P := \{x \in \mathbb{R}^n : (a_i^T x \geq b_i, \forall i \in I) \wedge (a_i^T x = b_i, \forall i \in E)\}$ be a non-empty polyhedron. Let $F \subseteq P$ and $F \neq \emptyset$. Let $A := \{a_i : i \in I \cup E\}$. Then F is a minimal face of P iff* $\dim(F) = n - \operatorname{rank}(A)$.

Proof. (TODO) □