Inverse of $x \mapsto x \ln x$

Eklavya Sharma

We want to bound the inverse of $x \mapsto x \ln x$, i.e., if $y = x \ln x$, then we want to lower and upper bound x by simple functions of y.

Our main result is that $x \in \Theta(y/\ln y)$. We also prove tighter non-asymptotic bounds.

1 Preliminaries

Lemma 1 (Log bounds [1])**.**

$$\forall x \in \mathbb{R}_{>0}, \frac{x-1}{x} \leq \ln x \leq x - 1.$$

Lemma 2. *Let $y = x \ln x$. Then $y > 0 \iff x > 1$ and $y \geq e \iff x \geq e$.*

Proof.

$$x \leq 1 \implies \ln x \leq 0 \implies y = x \ln x \leq 0.$$

$$x > 1 \implies \ln x > 0 \implies y = x \ln x > 0.$$

Therefore, $x \geq 1 \iff y \geq 0$.

$$x < e \implies \ln x < 1 \implies y = x \ln x < e.$$

$$x \geq e \implies \ln x \geq 1 \implies y = x \ln x \geq e.$$

Therefore, $x \geq e \iff y \geq e$. □

Theorem 3. *Let $x \geq 1$ and $y = x \ln x$. Then*

$$1 < \ell \leq x \leq u \implies \frac{y}{\ln u} \leq x \leq \frac{y}{\ln \ell}.$$

Proof.

$$\begin{aligned} &\ell \leq x \leq u \implies \ln \ell \leq \ln x \leq \ln u \\ &\implies x \ln \ell \leq y \leq x \ln u \implies \frac{y}{\ln u} \leq x \leq \frac{y}{\ln \ell} \end{aligned}$$ □

The above theorem is useful because it helps us *refine* the bounds that we find.

2 Bounds when $y \geq e$

Theorem 4. *Let $y \geq e$ and $y = x \ln x$. Then $x \leq y$.*

Proof. By Lemma 2, $y \geq e \iff x \geq e$. $x \geq e \implies \ln x \geq 1 \implies y \geq x$. □

Theorem 5. *Let $x \geq e$ and $y = x \ln x$. Then $x \geq y/(\ln y)$.*

Proof. Set $u = y$ and use Theorems 3 and 4. □

Theorem 6. *Let $y > 1$ and $y = x \ln x$. Then*

$$x \leq \frac{e+1}{e} \frac{y}{\ln y}.$$

Proof. By Lemma 2, $x > 1$.

$$\frac{x \ln y}{y} = \frac{x(\ln x + \ln \ln x)}{x \ln x} = 1 + \frac{\ln \ln x}{\ln x}$$

Let $t = \ln x$. Then $t > 0$ and $(x \ln y)/y = 1 + (\ln t)/t$. Define $g(t) = (\ln t)/t$. Then

$$g'(t) = \frac{1 - \ln t}{t^2}$$

$g'(t)$ is positive for $t < e$, negative for $t > e$ and $g'(e) = 0$. Therefore, $g(t)$ is maximized at $t = e$, and the maximum value is $g(e) = 1/e$. Therefore,

$$\frac{x \ln y}{y} = 1 + g(\ln x) \leq 1 + \frac{1}{e} \implies x \leq \frac{e+1}{e} \frac{y}{\ln y}$$ □

3 Bounds when $y \geq 0$

Theorem 7. *Let $y = x \ln x$. Then $x \leq y + 1$.*

Proof. $\frac{x-1}{x} \leq \ln x \implies x - 1 \leq y$. □

Define

$$\ell(y) = \begin{cases} y/\ln(y+1) & y \neq 0 \\ 1 & y = 0 \end{cases}.$$

Note that $\ell(y)$ is continuous over $y \in (-1, \infty)$.

Theorem 8. *Let $x \geq 1$ and $y = x \ln x$. Then $\ell(y) \leq x$.*

Proof. This is true for $x = 1$. For $x > 1$, set $u = y + 1$ and use Theorems 3 and 7. □

Theorem 9. *Let $x \geq 1$ and $y = x \ln x$. Then $x \leq 2\ell(y) - 1$.*

Proof. This is true for $x = 1$, so let $x > 1$. By Lemma 2, $y > 0$.

$$x \leq 2\ell(y) - 1 \iff \frac{2y}{\ln(y+1)} \geq x + 1 \iff \frac{2x \ln x}{x+1} \geq \ln(x \ln x + 1)$$

Define $g(x)$ as

$$g(x) = \frac{2x \ln x}{x+1} - \ln(x \ln x + 1).$$

To prove that $x \leq 2\ell(y) - 1$, we need to prove that $g(x) \geq 0$.

Note that $g(1) = 0$. If we prove that $g'(x) \geq 0$ for all $x \geq 1$, then that would imply $g(x) \geq 0$ for all $x \geq 1$.

$$\begin{aligned} g'(x) &= 2\left(\frac{\ln x + 1}{x+1} - \frac{x \ln x}{(x+1)^2}\right) - \frac{\ln x + 1}{x \ln x + 1} \\ &= \frac{x^2(\ln x - 1) + 2x(\ln x)^2 + \ln x + 1}{(x+1)^2(x \ln x + 1)} \end{aligned}$$

$$\begin{aligned} & x^2(\ln x - 1) + 2x(\ln x)^2 + \ln x + 1 \\ & \geq x^2\left(\frac{x-1}{x} - 1\right) + 2x\left(\frac{x-1}{x}\right)^2 + \frac{x-1}{x} + 1 \qquad \text{(since } x > 1 \text{ and by Lemma 1)} \\ & = x + \frac{1}{x} - 2 = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0 \end{aligned}$$

Hence, for $x \geq 1$, $g'(x) \geq 0 \implies g(x) \geq 0 \implies x \leq 2\ell(y) - 1$. □

Theorem 10. *Let $y = x \ln x$. Then $\ell(y) - 1 \leq x$.*

Proof. (TODO) □

References

[1] Eklavya Sharma. Theoremdep: Bound on log. URL: https://sharmaeklavya2.github.io/theoremdep/nodes/bounds/log-bound.html.