Linear Algebra Cheat Sheet for OR

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Abstract

This document contains a list of all linear algebra results that I needed for my Operations Research coursework at UIUC. The target audience of this document is Operations Research students. It can be helpful to page these results into your brain before a linear-algebra-heavy exam/project, and proving these results yourself (except those marked DWAP) is a good exercise.

Notation:

- DWAP abbreviates "Don't Worry About the Proof".
- For any integer $n \geq 0$, define $[n] := \{1, 2, \ldots, n\}$.

Contents

1 Vector Spaces

1.1 Field

Definition 1 (Field). *A field is a set F equipped with two operations $+$ and $\times$ that satisfy some special properties. (you don't need to know the properties, unless you want to be a pure mathematician, in which case you can find them here).*
Every field has two special elements, called the additive identity (usually denoted by 0), and the multiplicative identity (usually denoted by 1).

Theorem 1 (DWAP). *The following are fields:* $\mathbb{Q}$ *(the set of rational numbers),* $\mathbb{R}$ *(the set of real numbers),* $\mathbb{C}$ *(the set of complex numbers).*

Theorem 2 (DWAP). $\mathbb{Z}_p := \{0, 1, \ldots, p-1\}$ *is a field when* p *is prime, where* $+$ *and* $\times$ *are addition and multiplication modulo* p.

1.2 Vector Space

Definition 2. *A vector space* V *over a field* F *is a set with a vector addition operation* $(V \times V \mapsto V)$ *and a scalar multiplication operation* $(F \times V \mapsto V)$ *which satisfies some special properties. (you don't need to know the properties, unless you want to be a pure mathematician, in which case you can find them here).*
Every vector space has a special vector, called the additive identity, denoted by $\mathbf{0}$. *The elements of* V *are called vectors. The elements of* F *are called scalars.*

Definition 3 (Subspace). *Let* V *be a vector space.* U *is called a subspace of* V *if* $U \subseteq V$ *and* U *is also a vector space.*

Theorem 3 (DWAP). *The set of all polynomials over field* F *forms a vector space.*

Theorem 4 (DWAP). *For a field* F, F^d *is a vector space.*

Definition 4 (Linear and affine combinations). *Let* V *be a vector space over field* F. *Let* $X := \{x^{(i)} : i \in [k]\}$, *where* $x^{(i)} \in V$. *Let* $y = \sum_{i=1}^{k} \alpha_i x^{(i)}$, *where* $\alpha_i \in F$.

- y *is a called a* linear combination *of* X.
- *If* $\sum_{i=1}^{k} \alpha_i = 1$, *then* y *is called an* affine *combination of* X.

Definition 5 (Span). $\operatorname{span}(X)$ *is defined as the set of all linear combinations of* X. *For sets* X *and* Y *of vectors,* X *is called a spanning set of* Y *if* $Y \subseteq \operatorname{span}(X)$.

Lemma 5. *If* X *spans* S, *then* X *also spans* $\operatorname{span}(S)$.

Theorem 6 (DWAP). *Let* X *be a finite subset of* F^d, *where* F *is a field. Then* $\operatorname{span}(X)$ *is a vector space.*

1.3 Linear Independence and Basis

Definition 6 (Linear independence). *A set* $\{x_1, x_2, \ldots, x_n\}$ *of vectors over field* F *is called* linearly independent *iff*

$$\forall(\alpha_1, \ldots, \alpha_n) \in F^n, \left(\sum_{i=1}^{n} \alpha_i x_i = 0 \implies (\alpha_i = 0 \; \forall i \in [n]) \right).$$

Lemma 7 (Incrementing a linearly independent set). *Let* X *be a linearly independent set of vectors and* y *be a vector. If* $y \notin \operatorname{span}(X)$, *then* $X \cup \{y\}$ *is linearly independent.*

Lemma 8 (Decrementing a linearly dependent set). *Let* X *be a linearly dependent set of vectors. Then* $\exists x \in X$ *such that* $\operatorname{span}(X) = \operatorname{span}(X - \{x\})$.

Theorem 9 (DWAP)**.** *Let X be a spanning set of vector space V. If $Y \subseteq V$ and $|Y| > |X|$, then Y is linearly dependent.*

Definition 7 (Basis)**.** *Let S be a subset of vector space V. Then $X \subseteq S$ is a basis of S iff (the following definitions are equivalent):*

- *X is linearly independent and spans S.*
- *X is the largest linearly independent subset of S.*
- *X is a maximal linearly independent subset of S.*
- *X is the smallest spanning subset of S.*
- *X is a minimal spanning subset of S.*

Equivalence of these definitions can be proven using Theorem 9 and Lemmas 7 and 8.

Lemma 10. *If X is a basis of S, then X is also a basis of* $\operatorname{span}(S)$.

Lemma 11. *Let F be a field. Let $e^{(i)} \in F^d$ be a vector whose i^{th} coordinate is 1 and other coordinates are 0. Then $E := \{e^{(i)} : i \in [d]\}$ is a basis of F^d. (E is called the* standard basis *of F^d.)*

Theorem 12. *All bases of S have the same size. This size is called the rank of S (denoted as* $\operatorname{rank}(S)$*). If S is a vector space, it's called the dimension of S (denoted as* $\dim(S)$*).*

Theorem 13. *Let X be a set of* $\operatorname{rank}(S)$ *vectors. Then*

$$X \text{ is a basis of } S \iff X \text{ is linearly independent } \iff X \text{ spans } S.$$

Theorem 14 (Coordinatization)**.** *Let $B := \{b^{(1)}, b^{(2)}, \ldots, b^{(k)}\}$ be a basis of vector space V. Then $\forall x \in V$ there is a unique tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $x = \sum_{i=1}^{k} \alpha_i b^{(i)}$.*

1.4 Elementary Operations

Definition 8. *For a sequence $X := [x_1, \ldots, x_n]$ of vectors (over field F), an elementary operation is one of the following:*

- *For $i \neq j$ and $\alpha \in F$, replace x_i by $x_i + \alpha x_j$.*
- *For $i \in [n]$ and $\alpha \in F - \{0\}$, replace x_i by αx_i.*
- *For $i \neq j$, swap x_i and x_j.*

Lemma 15 (Reversibililty)**.** *Let X be a set of vectors. Let Y be the vectors obtained by applying an elementary operation on X. Then X can be obtained by applying an elementary operation to Y.*

Lemma 16. *For a set X of vectors, applying elementary operations doesn't change* $\operatorname{span}(X)$ *or* $\operatorname{rank}(X)$.

1.5 Affine Independence

Definition 9. *A set* $\{x_1, x_2, \ldots, x_n\}$ *of vectors over field* F *is called* affinely independent *iff*

$$\forall(\alpha_1, \ldots, \alpha_n) \in F^n, \left(\left(\sum_{i=1}^{n} \alpha_i = 0 \text{ and } \sum_{i=1}^{n} \alpha_i x_i = 0\right) \implies (\alpha_i = 0 \; \forall i \in [n])\right).$$

Theorem 17. *The set* $\{x^{(i)} : i \in [n]\}$ *of vectors is affinely independent iff* $\{x^{(i)} - x^{(n)} : i \in [n-1]\}$ *is linearly independent.*

Theorem 18. *The set* $\{x^{(i)} : i \in [n]\}$ *of vectors from* F^d *is affinely independent iff* $\{(x^{(i)}, 1) : i \in [n]\}$ *is linearly independent.*

2 Matrices

Lemma 19 (Matrix of elementary rowops)**.** *For a sequence* S *of elementary operations, there is a unique matrix* R *such that applying* S *to rows of any matrix* A *gives us* RA*.*

Definition 10 (row space, column space)**.** *Let* F *be a field and* $A \in F^{m \times n}$ *be a matrix, Let* $\text{rows}(A)$ *be the set of all row vectors of* A*, and* $\text{cols}(A)$ *be the set of all column vectors of* A*. Then*

- $\text{rowSpace}(A) := \text{span}(\text{rows}(A))$,
- $\text{colSpace}(A) := \text{span}(\text{cols}(A))$,
- $\text{rank}(A) := \text{rowRank}(A) := \text{rank}(\text{rows}(A))$,
- $\text{colRank}(A) := \text{rank}(\text{cols}(A))$.

Theorem 20 (DWAP)**.** *For any matrix* A*,* $\text{rowRank}(A) = \text{colRank}(A)$*.*

Definition 11 (Nullspace and nullity)**.** *For a matrix* $A \in F^{m \times n}$*,* $\text{nullSpace}(A) := \{x \in F^n : Ax = 0\}$ *and* $\text{nullity}(A) := \dim(\text{nullSpace}(A))$*.*

Theorem 21 (Rank-nullity theorem, DWAP)**.** $\text{rank}(A) + \text{nullity}(A) = |\text{cols}(A)|$*.*

Proof sketch. We can show that row space and nullspace are not affected by elementary row operations on A. Hence, we can assume that A is in Reduced-Row Echelon Form. There are $\text{rank}(A)$ pivot columns in A. Given any value of non-pivot variables, we can compute the value of pivot variables such that $Ax = 0$. Hence, $\text{nullity}(A) = |\text{cols}(A)| - \text{rank}(A)$. □

Theorem 22 (DWAP)**.** *If* V *is a subspace of* F^d*, then* $\exists A$ *such that* $V = \text{nullSpace}(A)$*.*

Theorem 23 (DWAP)**.** *Basic results on matrix multiplication:*

- *Matrix multiplication is associative, i.e.,* $(AB)C = A(BC)$*.*
- $(AB)^T = B^T A^T$*.*
- $|AB| = |A||B|$ *if* A *and* B *are square (*$|A|$ *is the determinant of* A*).*
- $(AB)^{-1} = B^{-1} A^{-1}$ *if* A *and* B *are invertible.*

Theorem 24 (Matrix singularity, DWAP)**.** *Let $A \in F^{n \times n}$. The following are equivalent:*

- $\text{rank}(A) = n$.
- *0 is the unique solution to $Ax = 0$.*
- $|A| \neq 0$ *($|A|$ is the determinant of A).*
- *A is invertible, i.e., $\exists B \in F^{n \times n}$ such that $AB = BA = I$.*

Theorem 25. *Let $A \in F^{n \times n}$. Then*

$$(A^{-1})[i, j] = \frac{(-1)^{i+j}}{|A|} |A[[n] - \{j\}, [n] - \{i\}]|.$$

Corollary 25.1.

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}^{-1} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}.$$

3 Miscellaneous

Definition 12 (p-norms)**.** *For $x \in \mathbb{R}^d$,*

$$\|x\|_p := \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \qquad \|x\|_\infty := \max_{i=1}^{d} |x_i| \qquad \|x\| := \|x\|_2$$

Definition 13 (Linear combinations)**.** *Let V be a vector space over field $\mathbb{R}$. Let $X := \{x^{(i)} : i \in [k]\}$, where $x^{(i)} \in V$. Let $y = \sum_{i=1}^{k} \alpha_i x^{(i)}$, where $\alpha_i \in F$.*

- *y is a called a* linear combination *of X.*
- *If $\alpha_i \geq 0$ for all $i \in [k]$, then y is a called a non-negative linear combination of X.*
- *If $\sum_{i=1}^{k} \alpha_i = 1$, then y is called an* affine *combination of X.*
- *A non-negative affine combination is called a convex combination.*

Theorem 26 (Cauchy-Schwarz inequality)**.** *$\forall x, y \in \mathbb{R}^d$, $|x^T y| \leq \|x\| \|y\|$.*