Nash Equilibrium

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Definition 1. *A strategy profile s^* for game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is called a* pure strategy Nash equilibrium *(PSNE) iff*

$$\forall i \in N, u_i(s_i^*, s_{-i}^*) = \max_{a \in S_i} u_i(a, s_{-i}^*)$$

Equivalently, this means that for every player, unilateral deviations do not increase utility.

1 Examples

1.1 Coordination game

1 \ 2	A	B
A	100, 100	0, 0
B	0, 0	10, 10

Here (A, A) and (B, B) are PSNE but (A, B) and (B, A) are not PSNE.

1.2 Prisoner's dilemma

1 \ 2	C	B
C	$-2, -2$	$-10, -1$
B	$-1, -10$	$-5, -5$

Here (B, B) is the only PSNE.

2 Properties of PSNE

Theorem 1. *A very weak DSE is also a PSNE.*

Proof.

$$\begin{aligned}
& s^* \text{ is a very weak DSE} \\
\iff & \forall i \in N,\ \forall s_i \in S_i,\ \forall s_{-i} \in S_{-i},\ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \\
\implies & \forall i \in N,\ \forall s_i \in S_i,\ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \\
\iff & s^* \text{ is a PSNE}
\end{aligned}$$

□

Theorem 2. *If a game contains a strong DSE, then that is the only PSNE.*

Proof. Assume the game contains a strong DSE s^* and a PSNE t^* such that $s^* \neq t^*$. Since $s^* \neq t^*$, there exists a player i such that $s_i^* \neq t_i^*$.

$$
\begin{aligned}
& s^* \text{ is a strong DSE} \\
& \implies s_i^* \text{ is a strongly dominant strategy for player } i \\
& \iff \forall s_i \in S_i - \{s_i^*\},\ \forall s_{-i} \in S_{-i},\ u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \\
& \implies u_i(s_i^*, t_{-i}^*) > u_i(t_i^*, t_{-i}^*)
\end{aligned}
$$

$$
\begin{aligned}
& t^* \text{ is a PSNE} \\
& \implies \forall s_i \in S_i,\ u_i(t_i^*, t_{-i}^*) \geq u_i(s_i, t_{-i}^*) \\
& \implies u_i(t_i^*, t_{-i}^*) \geq u_i(s_i^*, t_{-i}^*)
\end{aligned}
$$

This is a contradiction, so we cannot have a strong DSE and a different PSNE together in a game. □