# Dominant Strategy Equilibria

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**Definition 1.** *For a player $i$,*

- *action $a$* strongly dominates *action $b$ iff*
  $u_i(a, s_{-i}) > u_i(b, s_{-i})$ *for all* $s_{-i} \in S_{-i}$.
- *action $a$* very weakly dominates *action $b$ iff*
  $u_i(a, s_{-i}) \geq u_i(b, s_{-i})$ *for all* $s_{-i} \in S_{-i}$.
- *action $a$* weakly dominates *action $b$ iff*
  $u_i(a, s_{-i}) \geq u_i(b, s_{-i})$ *for all* $s_{-i} \in S_{-i}$ *and*
  $u_i(a, s_{-i}) > u_i(b, s_{-i})$ *for some* $s_{-i} \in S_{-i}$.

**Definition 2.** *For a player $i$, action $a$ is a (strongly/weakly/very weakly)* ***dominant strategy*** *iff $a$ (strongly/weakly/very weakly) dominates all others actions in $S_i$.*

**Definition 3.** *A strategy profile $s^*$ is a (strongly/weakly/very weakly)* ***dominant strategy equilibrium*** *(DSE) iff for each player $i \in N$, $s_i^*$ is a (strongly/weakly/very weakly) dominant strategy.*

# 1 Examples

## 1.1 Prisoner's Dilemma

Prisoner's Dilemma is a 2-player game where each player can either cooperate ($C$) or betray ($B$). It has the following payoff matrix:

| 1 \ 2 | C | B |
|---|---|---|
| C | $-2, -2$ | $-10, -1$ |
| B | $-1, -10$ | $-5, -5$ |

**Theorem 1.** *$(B, B)$ is a strongly dominant strategy equilibrium.*

*Proof.* $u_1(C, C) = -2 < -1 = u_1(B, C)$ and $u_1(C, B) = -10 < -5 = u_1(B, B)$. Therefore, $B$ is a strongly dominant strategy for player 1. By symmetry, $B$ is also a strongly dominant strategy for player 2. Therefore, $(B, B)$ is a strong DSE. □

If we change the payoff matrix to the following, $(B, B)$ will become a weak DSE:

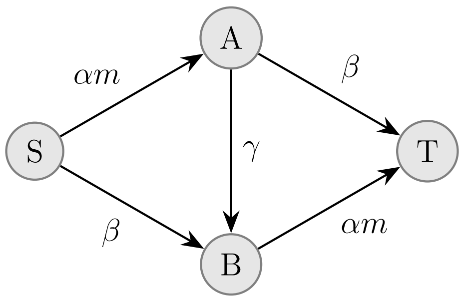
| 1 \ 2 | C | B |
|---|---|---|
| C | $-2, -2$ | $-10, -2$ |
| B | $-2, -10$ | $-5, -5$ |

If we change the payoff matrix to the following, all strategy profiles become very weak DSE, and no strategy profile is a weak DSE:

| 1 \ 2 | C | B |
|---|---|---|
| C | $-2, -2$ | $-5, -2$ |
| B | $-2, -5$ | $-5, -5$ |

## 1.2 Braess Paradox

Consider the following road network, where $n$ players wish to travel from $S$ to $T$, and each player wants to minimize the time taken to travel from $S$ to $T$.




The weight of an edge gives the time taken to traverse that edge. Here $m$ is the number of vehicles using that road, and $\alpha, \beta, \gamma$ are non-negative constants. The utility of a player is the negative of the time taken to travel from $S$ to $T$.

Each player has the strategy set $\{A, B, AB\}$ (corresponding to the paths $S \to A \to T$, $S \to B \to T$ and $S \to A \to B \to T$, respectively). For a strategy profile $s$ and action $X$, let $n_X(s)$ be the number of players who chose strategy $X$. Note that $n_A(s) + n_B(s) + n_{AB}(s) = n$. The utility function is given by

$$u_i(s) = -\begin{cases} \alpha(n_A(s) + n_{AB}(s)) + \beta & s_i = A \\ \beta + \alpha(n_B(s) + n_{AB}(s)) & s_i = B \\ \gamma + \alpha(n + n_{AB}(s)) & s_i = AB \end{cases}$$

**Theorem 2.** *$AB$ is a strongly dominant strategy for each player iff $\gamma < \beta - \alpha n$. $AB$ is a very weakly dominant strategy for each player iff $\gamma \leq \beta - \alpha n$. If $n \geq 3$ and $\alpha > 0$, then $AB$ is a weakly dominant strategy for each player iff $\gamma \leq \beta - \alpha n$.*

*Proof.* For any player, consider a strategy profile $s_{-i}$ for the other players. Then

$$\begin{aligned} &u_i(AB, s_{-i}) - u_i(A, s_{-i}) = (-u_i(A, s_{-i})) - (-u_i(AB, s_{-i})) \\ &= (\alpha(n_A(s_{-i}) + 1 + n_{AB}(s_{-i})) + \beta) - (\gamma + \alpha(n + n_{AB}(s_{-1}) + 1)) \\ &= (\beta - \gamma) + \alpha(n_A(s_{-i}) - n) \\ &\geq (\beta - \alpha n) - \gamma \end{aligned}$$

$$
\begin{aligned}
u_i(AB, s_{-i}) - u_i(B, s_{-i}) &= (-u_i(B, s_{-i})) - (-u_i(AB, s_{-i})) \\
&= (\beta + \alpha(n_B(s_{-i}) + 1 + n_{AB}(s_{-i}))) - (\gamma + \alpha(n + n_{AB}(s_{-1}) + 1)) \\
&= (\beta - \gamma) + \alpha(n_B(s_{-i}) - n) \\
&\geq (\beta - \alpha n) - \gamma
\end{aligned}
$$

Therefore, for player $i$, action $AB$ is a strongly dominant strategy if $\gamma < \beta - \alpha n$ and action $AB$ is a very weakly dominant strategy if $\gamma \leq \beta - \alpha n$.

Let $s^*_{-i}$ be the strategy profile where all players other than $i$ choose action $AB$. Then

$$u_i(AB, s^*_{-i}) - u_i(A, s^*_{-i}) = (\beta - \gamma) + \alpha(n_A(s^*_{-i}) - n) = (\beta - \alpha n) - \gamma$$

$$u_i(AB, s^*_{-i}) - u_i(B, s^*_{-i}) = (\beta - \gamma) + \alpha(n_B(s^*_{-i}) - n) = (\beta - \alpha n) - \gamma$$

Therefore, for player $i$, action $AB$ is a strongly dominant strategy iff $\gamma < \beta - \alpha n$ and action $AB$ is a very weakly dominant strategy iff $\gamma \leq \beta - \alpha n$.

Let $n \geq 3$, $\alpha > 0$ and $\gamma \leq \beta - \alpha n$. Let $\widehat{s}_{-i}$ be the strategy profile where $\lfloor (n-1)/2 \rfloor$ players from $N - \{i\}$ play strategy $A$ and $\lceil (n-1)/2 \rceil$ players from $N - \{i\}$ play strategy $B$. Then

$$u_i(AB, \widehat{s}_{-i}) - u_i(A, \widehat{s}_{-i}) = (\beta - \alpha n - \gamma) + \alpha n_A(\widehat{s}_{-i}) > 0$$

$$u_i(AB, \widehat{s}_{-i}) - u_i(B, \widehat{s}_{-i}) = (\beta - \alpha n - \gamma) + \alpha n_B(\widehat{s}_{-i}) > 0$$

Therefore, $AB$ is a weakly dominant strategy for player $i$. □

Let $n = 1000$, $\alpha = 1/50$, $\beta = 25$ and $\gamma = 0$. By Theorem 2, $(AB)_{i \in N}$ is a strong DSE. Then for each player, the utility of the DSE is $-(\gamma + 2\alpha n) = -40$. Let $s^*$ be the strategy profile where half the players play $A$ and the others play $B$. Then for each player $i$, $u_i(s^*) = -(\beta + \alpha n/2) = -35$. Therefore, the utility of $s^*$ is higher than that of the strong DSE.

## 1.3 Second-price Auction

Consider a second-price auction with $n$ players. Let $v_i$ and $b_i$ be the valuation and bid, respectively, of player $i$.

Let $y_i(b)$ be 1 iff player $i$ wins for the bid profile $b$ and 0 otherwise. Let $t(b)$ be the second-highest bid in $b$ (if there are multiple highest bids, they are also second-highest bids). Then $u_i(b) = y_i(b)(v_i - t(b))$.

**Lemma 3.** *For every player $i$, $b_i = v_i$ is a weakly dominant strategy.*

*Proof.* Consider any $b_i \neq v_i$. We will first show that $v_i$ very weakly dominates $b_i$. Let $b_{-i}$ be any bid profile of the other players.

For any $x \in \mathbb{R}$, we get

$$y_i(x, b_{-i}) = 1 \implies x \geq \max(b_{-i}) = t(x, b_{-i})$$

$$y_i(x, b_{-i}) = 0 \implies x \leq \max(b_{-i})$$

**Case 1a**: $y_i(v_i, b_{-i}) = 1$ and $y_i(b_i, b_{-i}) = 1$.

$$\implies t(v_i, b_{-i}) = \max(b_{-i}) \quad \text{and} \quad t(b_i, b_{-i}) = \max(b_{-i})$$

$$\implies u_i(v_i, b_{-i}) = v_i - t(v_i, b_{-i}) = v_i - \max(b_{-i}) = v_i - t(b_i, b_{-i}) = u_i(b_i, b_{-i})$$

**Case 1b**: $y_i(v_i, b_{-i}) = 1$ and $y_i(b_i, b_{-i}) = 0$.

$$\implies v_i \geq t(v_i, b_{-i}) = \max(b_{-i})$$

$$\implies u_i(v_i, b_{-i}) = v_i - t(v_i, b_{-i}) \geq 0 = u_i(b_i, b_{-i})$$

**Case 2a**: $y_i(v_i, b_{-i}) = 0$ and $y_i(b_i, b_{-i}) = 1$.

$$\implies v_i \leq \max(b_{-i}) = t(b_i, b_{-i}) \leq b_i$$

$$\implies u_i(v_i, b_{-i}) = 0 \geq v_i - t(b_i, b_{-i}) = u_i(b_i, b_{-i})$$

**Case 2b**: $y_i(v_i, b_{-i}) = 0$ and $y_i(b_i, b_{-i}) = 0$.
Then $u_i(v_i, b_{-i}) = 0 = u_i(b_i, b_{-i})$.

Since $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all $b_{-i}$, $v_i$ very weakly dominates $b_i$.

We will now show that $v_i$ weakly dominates $b_i$. Consider the profile $b_{-i}$ where all players other than $i$ bid $(v_i + b_i)/2$.

**Case 1**: $b_i > v_i$.
Then player $i$ wins with bid $b_i$ and loses with bid $v_i$, so $u_i(v_i, b_{-i}) = 0$ and

$$u_i(b_i, b_{-i}) = v_i - \frac{v_i + b_i}{2} = \frac{v_i - b_i}{2} < 0.$$

Therefore, $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$.

**Case 2**: $v_i > b_i$.
Then player $i$ wins with bid $v_i$ and loses with bid $b_i$, so $u_i(b_i, b_{-i}) = 0$ and

$$u_i(v_i, b_{-i}) = v_i - \frac{v_i + b_i}{2} = \frac{v_i - b_i}{2} > 0.$$

Therefore, $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$. □

**Corollary 3.1.** *For second-price auctions, the bid profile $(v_1, v_2, \ldots, v_n)$ is a weak DSE.*

# 2 Properties of DSE

**Lemma 4.** *For any player, if there are two distinct very weakly dominant strategies, then none of them is a weakly dominant strategy.*

*Proof.* Assume player $i$ has two distinct strategies $a$ and $b$ where $a$ is weakly dominant and $b$ is very weakly dominant. Since $a$ weakly dominates $b$, $\exists s_{-i} \in S_{-i}$ such that $u_i(a, s_{-i}) > u_i(b, s_{-i})$. Since $b$ very weakly dominates $a$, $u_i(b, s_{-i}) \geq u_i(a, s_{-i})$. This is a contradiction, so for any player, there cannot be a weakly dominant strategy and a different very weakly dominant strategy. □

**Theorem 5.** *If a strategic form game contains a weak DSE, then it does not contain any other very weak DSE.*

*Proof.* Assume there is a weak DSE $s$ and a very weak DSE $t$ such that $s \neq t$. Since $s \neq t$, there is a player $i$ such that $s_i \neq t_i$. Therefore, $s_i$ is a weakly dominant strategy for player $i$ and $t_i$ is a very weakly dominant strategy for player $i$. But this contradicts Lemma 4. □