Theorems

1 Abstract Algebra

Definition 1. *A monoid M is a set S along with a binary operator $\circ : S \times S \to S$ which satisfies these properties:*

1. Associativity*: $\forall a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$.*
2. Existence of Identity*: There exists an element $e \in S$, called an* identity *of S, such that $\forall a \in S$, $a \circ e = e \circ a = a$.*

Lemma 1. *Every monoid has a unique identity.*

Proof. Let $M = (S, \circ)$ be a monoid. Let e_1 and e_2 be any two (possibly identical) identities of M. Then $e_1 \circ e_2 = e_1$, because e_2 is an identity, and $e_1 \circ e_2 = e_2$, because e_1 is an identity. Hence, $e_1 = e_2$. Hence, M has a unique identity □

2 Convexity

Definition 2. *For a finite set $P := \{x_1, \ldots, x_n\}$, the* convex hull *of P is defined as*

$$\mathrm{conv}(P) := \left\{ \sum_{i=1}^{n} \alpha_i x_i \;\middle|\; \alpha_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Theorem 2 (Carathéodory's theorem [1])**.** *Let $x \in \mathrm{conv}(P)$, where $P \subseteq \mathbb{R}^d$ is a finite set. Then $x \in \mathrm{conv}(Q)$, where $Q \subseteq P$ and $|Q| \leq d + 1$.*

References

[1] Constantin Carathéodory. Über den variabilitätsbereich der fourier'schen konstanten von positiven harmonischen funktionen. *Rendiconti Del Circolo Matematico di Palermo (1884-1940)*, 32(1):193–217, 1911. doi:10.1007/BF03014795.