Oral Qualifying Exam

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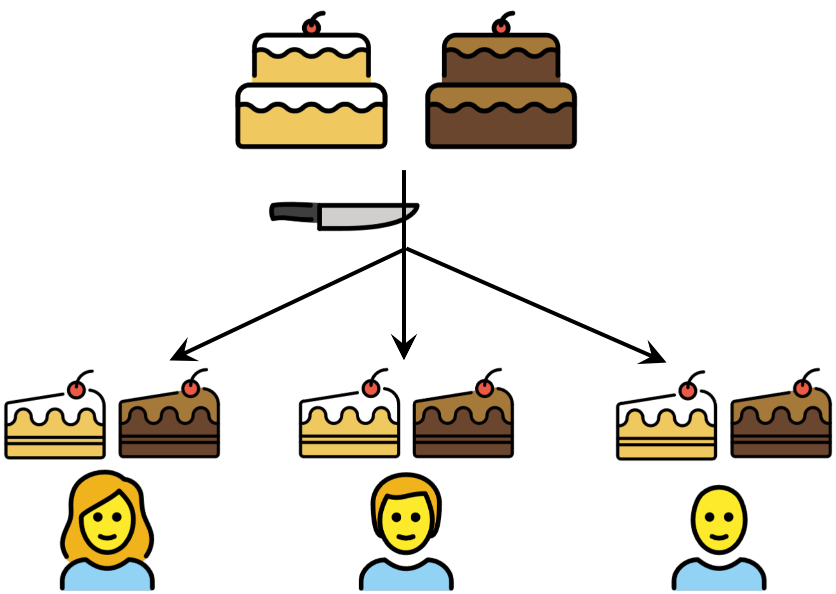
The Paper

- Best of Both Worlds: Ex Ante and Ex Post Fairness in Resource Allocation [doi:10.1287/opre.2022.2432]
- by Haris Aziz, Rupert Freeman, Nisarg Shah, and Rohit Vaish,
- in Operations Research (INFORMS), Jan 2023.
- Preliminary version in Conference on Economics and Computation, July 2020.

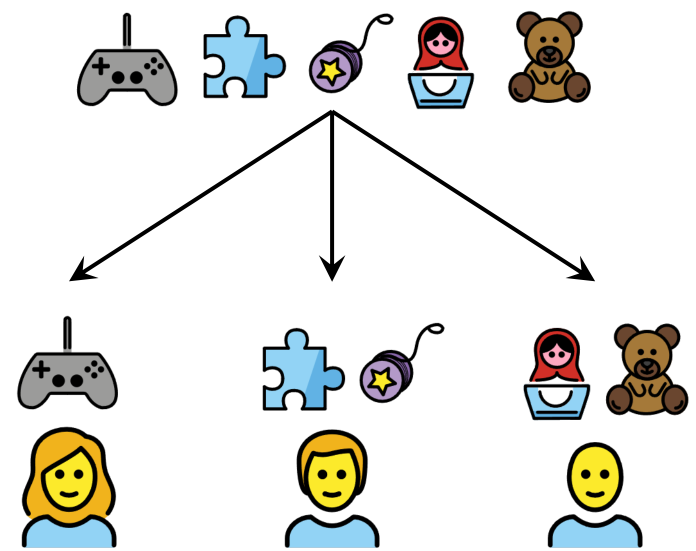
Fair Division of Goods

Divide goods among n people (called agents), who are all ‘equally deserving’.

Divisible (continuous version):



Indivisible (discrete version):



Some Applications

- Business partnership dissolution / divorce
- Dividing radio frequency spectrum among communication companies.
- Dividing airplane runway time among airlines.

Formalizing the Problem

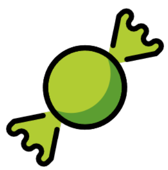
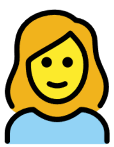
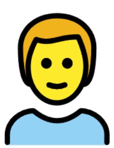
- n agents: 1 to n. m goods: 1 to m.
- Find *allocation* $A \in [0,1]^{n \times m}$ where $A_{i,j}$ is the fraction of good j allocated to agent i.
  - Each column sums to 1.
  - If goods are indivisible, A must be integral.
- A_i (i^{th} row of A) is called agent i's *bundle*.
  - For indivisible goods, A_i is like a subset of goods.

Valuations and Fairness

- Input: $v_{i,j}$ is agent i's value for good j (≥ 0).
- For bundle $x \in [0,1]^m$, let $v_i(x) = \sum_{j=1}^{m} x_j v_{i,j}$.
  - v_i is called agent i's valuation function.
- In allocation A, agent i *envies* agent j if $v_i(A_i) < v_i(A_j)$.
- Allocation A is *envy-free* (EF) if no one envies anyone.

Envy-Freeness: Example

A is envy-free (EF) if no one envies anyone else.

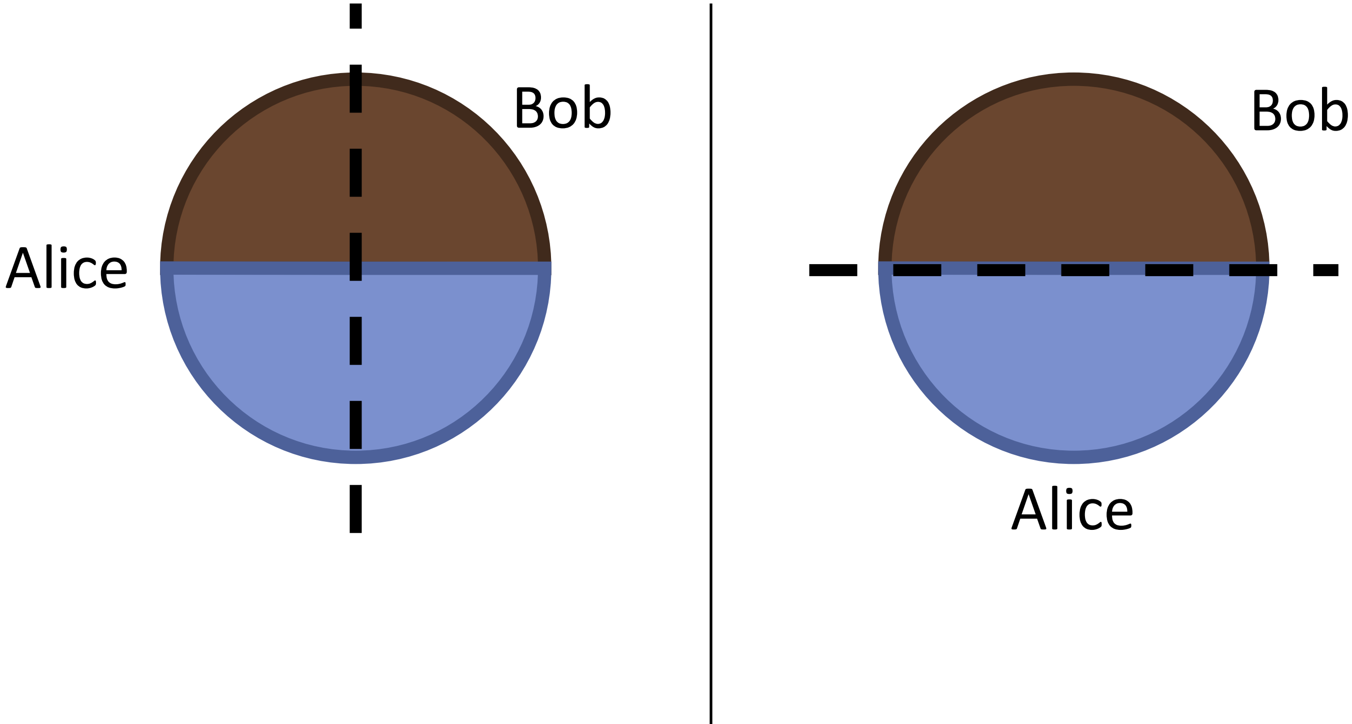
V			
	4	2	6
	5	15	25

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is EF.

EF allocations always exist for divisible goods:
for $A_{i,j} = 1/n$, A is EF.

Efficiency

- Fairness isn't the only concern.
- Alice prefers blueberry and Bob prefers chocolate.
- Both allocations are fair. But one is better.



Pareto Optimality (PO)

- Intuitively, an allocation is pareto optimal (PO) if it's impossible to make someone happier without making someone else sadder.
- Allocation X pareto-dominates allocation Y iff both of the following are true:
  - no one prefers Y: $\forall i, v_i(X_i) \geq v_i(Y_i)$.
  - someone prefers X: $\exists i, v_i(X_i) > v_i(Y_i)$.
- Allocation X is pareto-optimal (PO) if it is not pareto-dominated by any other allocation.

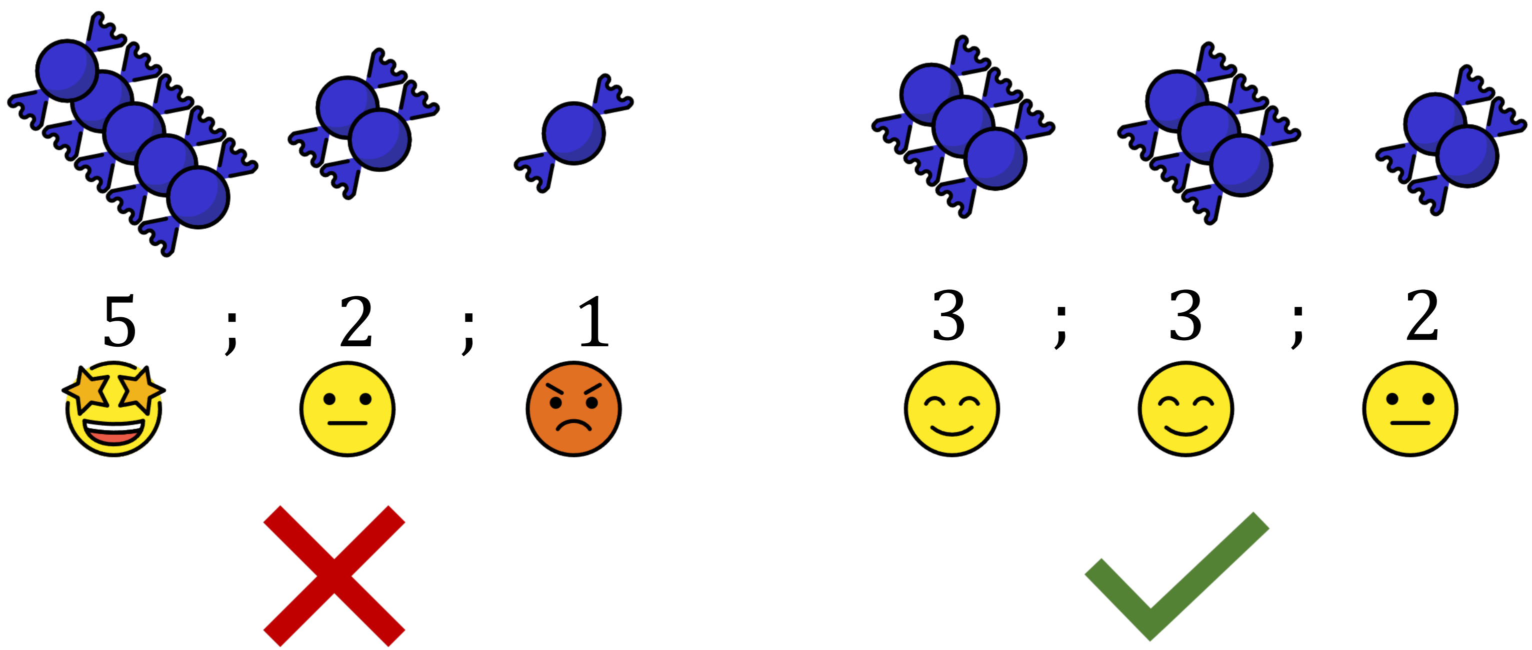
Nash Social Welfare (NSW)

- NSW is the 'average' happiness of an allocation.
- $\mathrm{NSW}(A) = \sqrt[n]{v_1(A_1)v_2(A_2)\cdots v_n(A_n)}$.
- An alloc that maximizes NSW is Nash Optimal.
- NashOpt implies PO.
- For divisible goods:
  - NashOpt is EF [1].
  - NashOpt allocations can be found in polytime [HME CH14, EG59].

Fairness for Indivisible Goods

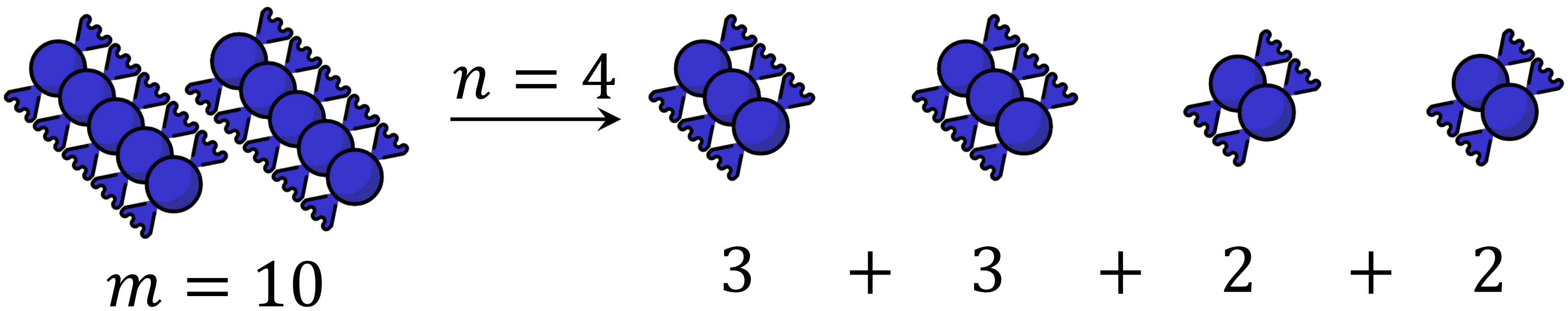
EF is not guaranteed

- For divisible goods, EF always exist.
- But not for indivisible goods: e.g., single good.
- We can’t be fair, but we can be *approximately* fair.



Fairness for the Indivisible setting

- Suppose there are m identical goods and n agents.
- Each agent should get $\lfloor m/n \rfloor$ or $\lceil m/n \rceil$ goods.
- How do we generalize this idea?



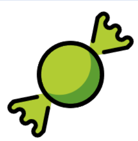
- Observation: $\lceil m/n \rceil - \lfloor m/n \rfloor \leq 1$.

EF1 (EF up to 1 good)

- Agent i is EF1-satisfied by allocation A if for $\forall j \neq i$, after removing some good from A_j, agent i no longer envies agent j.

$$\forall j \neq i, \exists g \in A_j, v_i(A_i) \geq v_i(A_j \setminus \{g\})$$

- An allocation is EF1 if all agents are EF1-satisfied.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		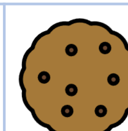	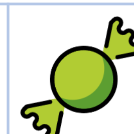	
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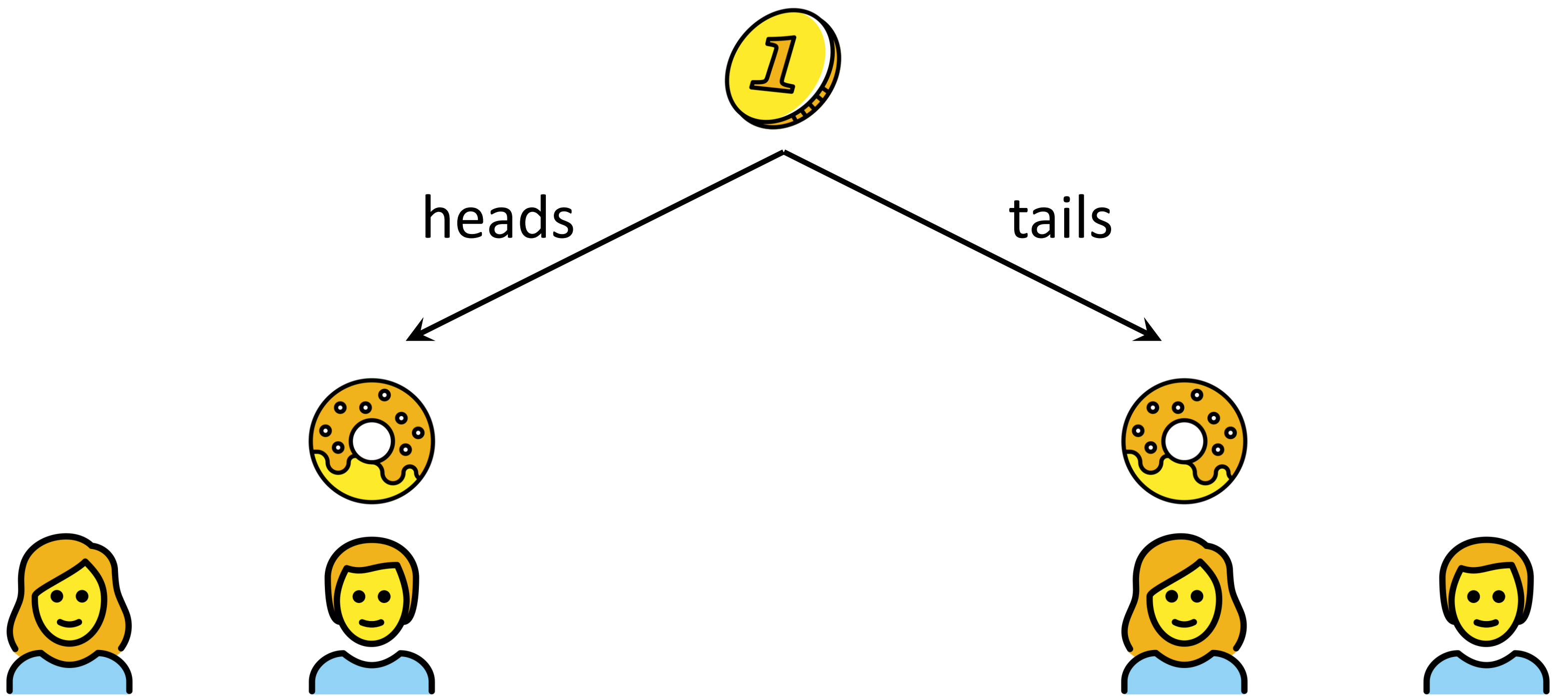
Indivisible goods: F&E

- EF1 allocations (unlike EF) always exist.
- EF1 allocations can be computed in $O(mn \log m)$ time using the Round-Robin algorithm.
- NashOpt allocations are EF1+PO. [EC'16]
- Computing NashOpt allocations is NP-hard.
- No known algorithm for computing EF1+PO allocations in polynomial time
  (but we can do it in pseudo-polynomial time [EC'18].)

Round-Robin Algorithm

- Let $r = \lceil m/n \rceil$ be the number of rounds.
- In each round,
  - agent 1 picks their good,
  - then agent 2 picks their favorite remaining good,
  - then agent 3 picks their favorite remaining good, …
- This is EF1. Why?

Random Allocations



Fairness of distributions

- A *distribution of allocations* is a set $\{(p_k, A^{(k)}) : k \text{ from } 1 \text{ to } K\}$, where
  - $A^{(k)}$ is an integral allocation.
  - $\sum_{k=1}^{K} p_k = 1$ and $p_k \geq 0 \; \forall k$.
  - We pick allocation $A^{(k)}$ with probability p_k.
- $\bar{A} = \sum_{k=1}^{K} p_k A^{(k)}$ is called the distribution's *mean*.
- $\bar{A}_{i,j} = q$: agent i gets good j with probability q.
- Equivalently, $\bar{A}_i$ is agent i's *expected* bundle.
- A distribution is called *ex ante EF* if its mean is EF.

Ex ante EF is not sufficient

- Pick an agent urandomly and give all goods to her. This is EX ante EF.
- EX ante EF assumes agents are not risk-averse.
- Example: 4 identical goods and 3 agents:
  - Bad: ($^1/_3$, [4,0,0]), ($^1/_3$, [0,4,0]), ($^1/_3$, [0,0,4]).
  - Good: ($^1/_3$, [2,1,1]), ($^1/_3$, [1,2,1]), ($^1/_3$, [1,1,2]).
- A distribution $\{(p_k, A^{(k)}): k \in [K]\}$ is called *ex post EF1* if $A^{(k)}$ is EF1 for all k.
- Can we get ex ante EF and ex post EF1 together?

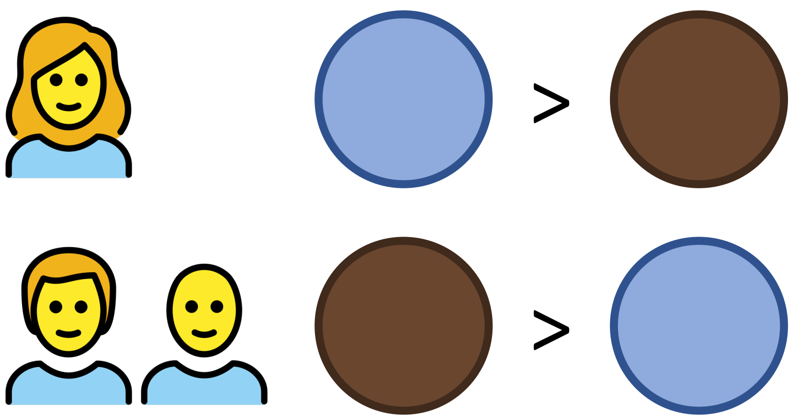
Paper's results

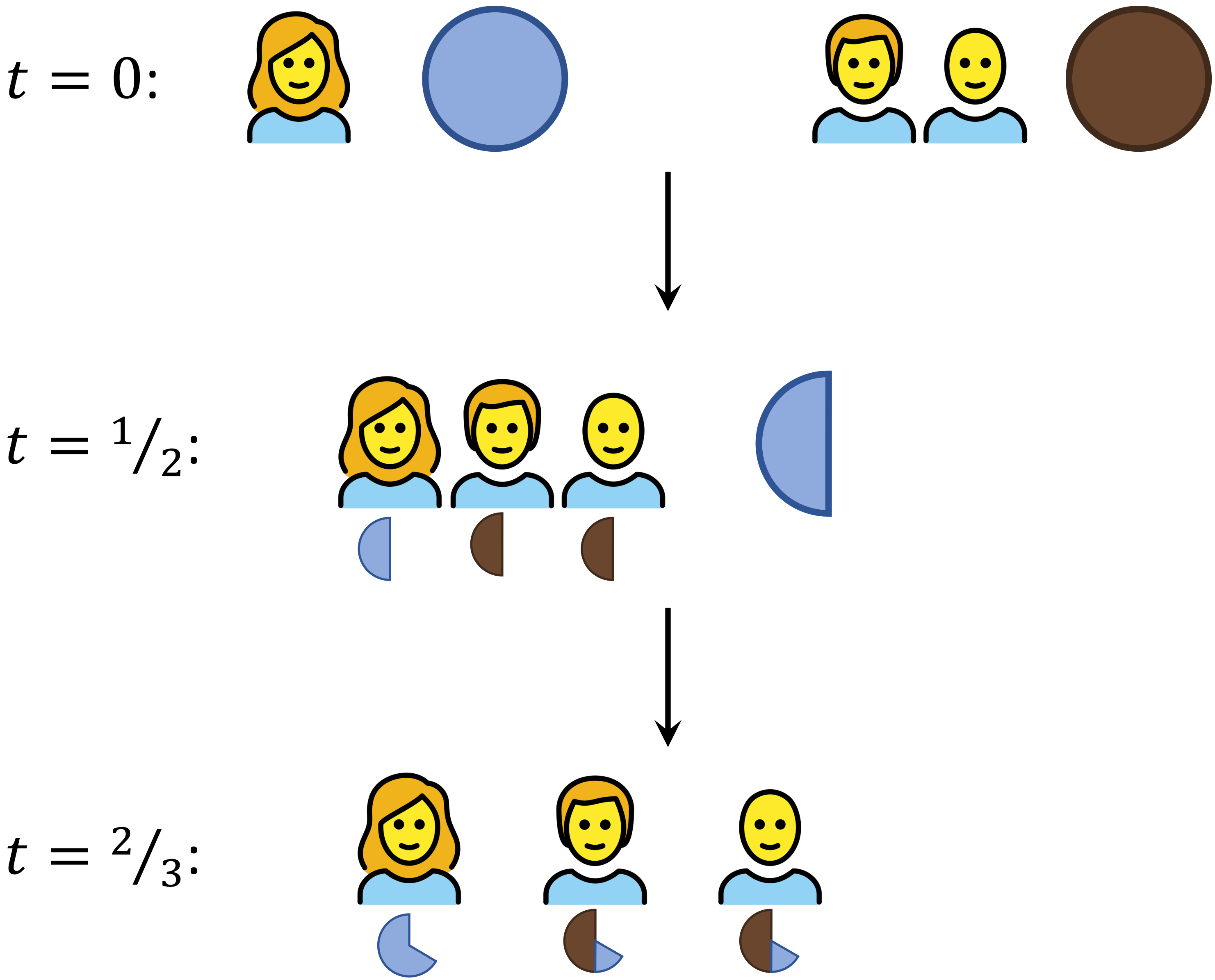
- Primary result:
  - Ex ante EF + ex post EF1
  - support size $\leq (m+n-1)^2+1$.
  - runs in time $O((m+n)^{9/2})$.
- Secondary result:
  - ex post PROP1 + ex post EF1MaL + Ex ante NashOpt ($\Longrightarrow$ ex ante EF + ex ante PO + ex post fPO).
  - strongly polynomial running time.
- Impossibility result: Example with $n = m = 2$ for which no allocation is ex ante EF + ex post EF1 + ex post fPO.

Probabilistic Serial (cont. RR)

Dividing m divisible goods among n agents:

- Goods are food items. All items have the same size.
- Agents start eating. All agents eat at the same rate.
- At any time, each agent eats her favorite good.




$t = 0$:
$t = {}^{1}/_{2}$:
$t = {}^{2}/_{3}$:

Ex ante EF + ex post EF1

- Each agent eats m/n goods.
- Output A is EF.
- Can we write A as a convex combination of integral EF1 allocations?
- Special case: $m = n$:
  - A is doubly stochastic.
  - By Birkhoff's theorem, A is a convex combination of permutation matrices.
  - Permutation matrices are integral EF1 allocations.

Conclusion

- Use of randomness in fair division was known, but it was unclear how to overcome the ‘give everything to random agent’ barrier.
- Paper started a line of works:
  - BoBW FairShare [WINE’22]:
    ex ante PROP + ex post ½-MMS + ex post PROP1.
  - BoBW with entitlements [AAMAS’23, arXiv]:
    ex ante NashOpt + ex post wEF1t.

Future Directions

- ex ante EF + ex post EF1 + ex post PO.
- Different ex post notions of fairness:
  EFX, EEFX, MMS, APS.
- Fair division of chores, mixed manna.
- Different valuation function classes.
- Different ways of handing the ‘give everything to random agent’ barrier.

Thank You

Questions?