Nash Equilibrium of Hand Cricket

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Hand Cricket and RUC Games

Hand cricket is a two-player game played with hand gestures (like rock-paper-scissors). It is popular among children in India.

We generalize it to *Repeat-Until-Collision (RUC) games*. Parametrized by matrices $A, B \in \mathbb{R}^{n \times n}_{\geq 0}$. Rules:

1. There are two players: max player (aka batter) and min player (aka bowler).
2. There are multiple rounds. In each round, both players simultaneously pick a number from $\{1, 2, \ldots, n\}$.
3. If max player picks i and min player picks j, max player scores $A[i, j]$, min player incurs a cost of $B[i, j]$.
4. If $i = j$ (collision), the game ends. Else, proceed to next round.
5. Max player wants to maximize her (expected) total score. Min player wants to minimize her (expected) total cost.

Hand cricket: $A[i, j] = B[i, j] = i$ when $i \neq j$ and 0 otherwise.

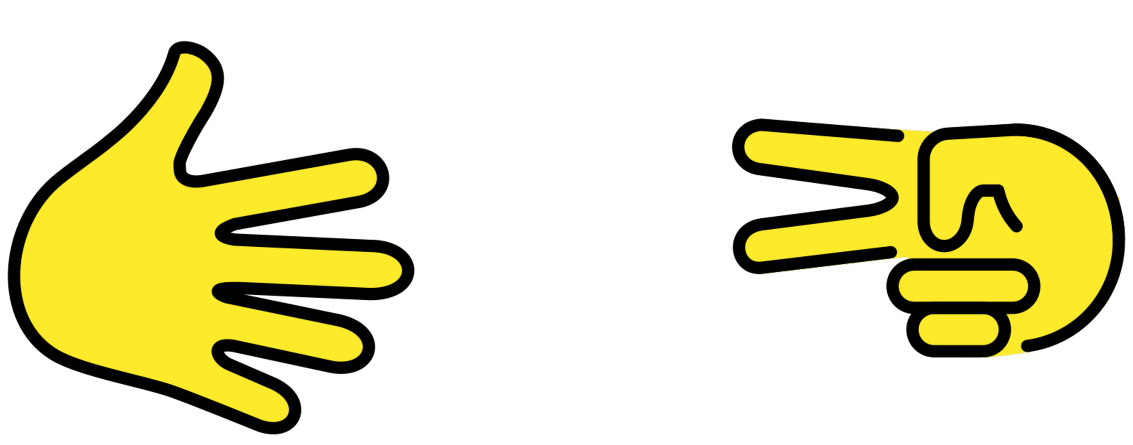

Figure 1. Single round of hand cricket, where max player (left) played 5 and min player (right) played 2. Max player earned 5 points. Min player incurred a cost of 5 points. $5 \neq 2$, so game doesn't end.

Min player wants the game to end soon to prevent accumulating a large cost. Max player wants the opposite.

Max player can score more per round by playing high-scoring actions more frequently, but then the min player can cause a collision sooner. Playing *optimally* requires carefully balancing this tradeoff.

Pursuit-Evasion Games

There are n locations. Each day, a drug dealer picks a location to sell drugs, and law enforcement picks a location for a random check. If the locations match, the dealer is caught and the game ends.

This is another example of an RUC game. They are also called *hide-and-seek* games.

Nash Equilibria

A pair (x, y) of strategies if called a *Nash Equilibrium* (NE) if both of these hold:

1. If min player plays y, then playing x maximizes max player's score.
2. If max player plays x, then playing y minimizes min player's cost.

Our Results

Let $\Delta_n := \{x \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\}$.

Stationary strategy $x \in \Delta_n$: pick each action i with probability x_i independently in each round.

A *stationary RUC (SRUC) game* is one where both players are forced to play only stationary strategies.

Theorem 1. Let A and B be irreducible matrices.

1. $\exists x, y \in \Delta_n$ such that (x, y) is an NE for both the SRUC game and the RUC game.
2. NE is unique for SRUC game iff $\operatorname{graph}(A) \subseteq \operatorname{graph}(B)$.
3. NE is almost-unique for RUC game if $A = B$.

Proof Sketch

SRUC Games

Theorem 2. If max player plays $x \in \Delta_n$ and min player plays $y \in \Delta_n$, then max player's expected total score is $x^T Ay/x^T y$ and min player's expected total cost is $x^T By/x^T y$.

- If y is A's eigenvector, max player's score is independent of x.
- If x is B^T's eigenvector, min player's cost is independent of y.

If such eigenvectors exist, they would give us NE. But do they exist? Yes!

Perron-Frobenius Theorem. Let $A \in \mathbb{R}^{n \times n}_{\geq 0}$ be irreducible. Then

1. *(Perron root)* $\exists$ eigenvalue $\rho \in \mathbb{R}_{\geq 0}$ that is largest in absolute value among all (complex) eigenvalues.
2. *(Perron vectors)* $\exists$ unique vectors u and v s.t. $A^T u = \rho u$, $Av = \rho v$, and $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i = 1$.
3. $u_i > 0$ and $v_i > 0$ for all i.

Example: If $A = \begin{bmatrix} 0 & s_1 \\ s_2 & 0 \end{bmatrix}$, then $\rho = \sqrt{s_1 s_2}$, $u = (1 - \alpha, \alpha)$, and $v = (\alpha, 1 - \alpha)$, where $\alpha = \dfrac{\sqrt{s_1}}{\sqrt{s_1} + \sqrt{s_2}}$.

RUC Games

Our proof exploits the recursive structure of RUC games: after the first round, if the game doesn't end, the remaining game is identical to the original.

Paper

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