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Fair Allocation

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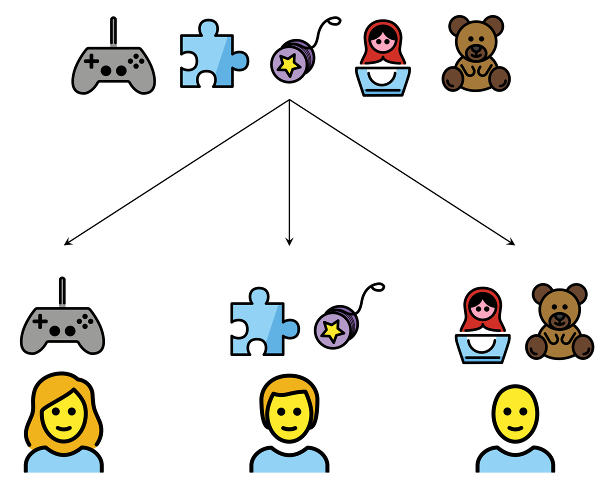
The Fair Division Problem

Distribute (indivisible) goods among agents *fairly*.

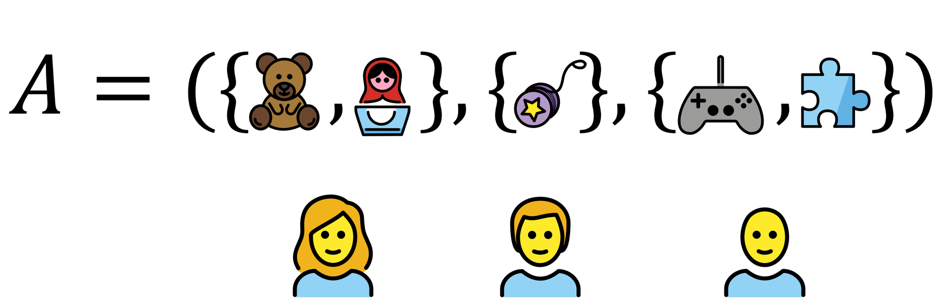


Input: n agents and a set M of m goods.
$v_i(g)$ is agent i's value for good g.
For any $S \subseteq M$, let $v_i(S) := \sum_{g \in S} v_i(g)$.

m (goods) × n (agents):

agent 1	10	4	8	9	3
agent 2	5	3	3	8	2
agent 3	5	2	0	2	2

Output: an alloction $A = (A_1, \ldots, A_n)$, where A_i is agent i's bundle of goods.



Defining Fairness

Envy-freeness (EF): Agent i is envy-free in allocation A if for every $j \neq i$, we have $v_i(A_i) \geq v_i(A_j)$.

Proportionality (PROP): Allocation A is PROP-fair to agent i if $v_i(A_i) \geq v_i(M)/n$. $v_i(M)/n$ is called i's PROP-share.

V				$\frac{v_i(M)}{n}$
agent 1	4	2	6	6
agent 2	5	15	25	22.5

Figure 1. An allocation that is both EF and PROP.

Theorem: If agent i is EF in allocation A, then A is PROP-fair to i.

EF or PROP allocations may not exist! E.g., if $m = 1$.

Approximate Fairness

Even though we can't be exactly fair, we can be approximately fair.

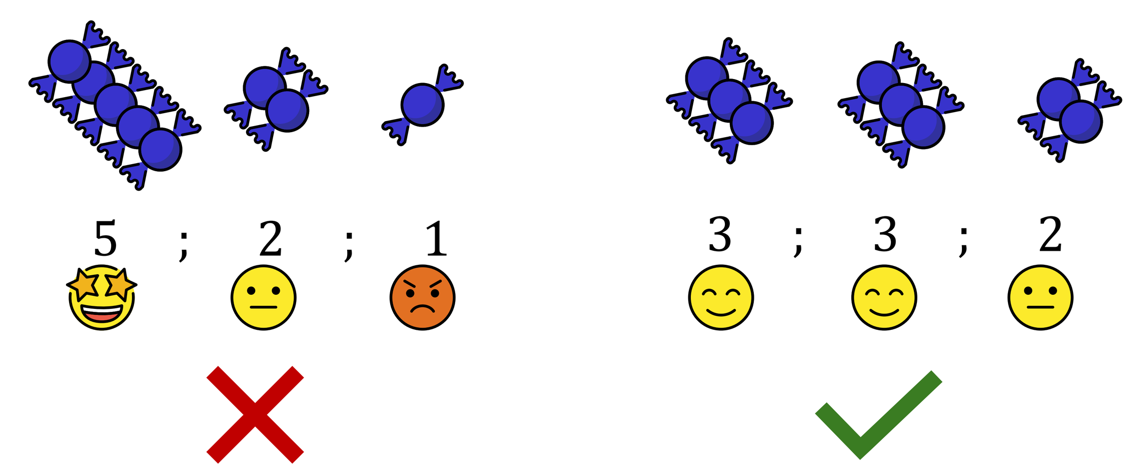


Figure 2. For $n = 3$ and $m = 8$, $(3, 3, 2)$ is fairer than $(5, 2, 1)$.

When goods are identical, each agent should get $\lfloor m/n \rfloor$ or $\lceil m/n \rceil$ goods. Equivalently, any two bundles should differ by at most 1 good. How do we generalize this?

Notions of (approximate) fairness of allocation A to agent i:

EFX: $\forall j \neq i, \forall g \in A_j, v_i(A_i) \geq v_i(A_j \setminus \{g\})$.

EF1: $\forall j \neq i$, either $A_j = \emptyset$ or $v_i(A_i) \geq v_i(A_j \setminus \{g\})$ for some $g \in A_j$.

MMS: $v_i(A_i) \geq \mu_i := \max_{(X_1,\ldots,X_n)} \min_{j=1}^{n} v_i(X_j)$.

EEFX: $\exists$ allocation B such that $B_i = A_i$ and B is EFX-fair to i.

For $n = 2$, EF = PROP and EFX = EEFX.

Allocation	EF1	EFX	MMS	EEFX
({🏠, 🚲}, {🚘})	✓	✗	✗	✗
({🏠}, {🚘, 🚲})	✓	✓	✓	✓

Figure 3. Example with $n = 2$ and $m = 3$.

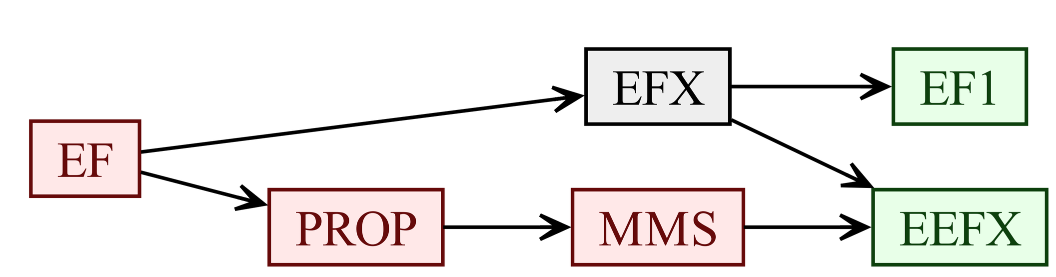


Figure 4. $F_1 \to F_2$ iff every F_1-fair allocation is also F_2-fair.

Key Problems

Feasibility: Does a fair allocation exist for every input?

Computability: Can we find a fair allocation in polynomial time if it exists?

Known Results

notion	feasible	computable	
EFX	*open*	*open*	
EF1	yes	polytime	[9]
EEFX	yes	polytime	[4]★
MMS	no	NP-hard	[5]
$3/4$-MMS	yes	polytime	[8]

For $n = 2$, EFX is feasible and polytime computable.

Introducing Randomness

Using randomization, we can get EF and PROP ***ex ante*** (i.e., in expectation).

E.g., for 3 agents and 7 identical goods, we output one of $(3, 2, 2)$, $(2, 3, 2)$, and $(2, 2, 3)$ with probability $1/3$ each.

Notions of randomized fairness of allocation A to agent i:
ex ante EF: $\forall j \neq i, \mathbb{E}(v_i(A_i)) \geq \mathbb{E}(v_i(A_j))$.
ex ante PROP: $\mathbb{E}(v_i(A_i)) \geq v_i(M)/n$.

	ex post	ex ante
[2]	EF1	EF
[3]	$1/2$-MMS	PROP
[6]	EF1 + $1/2$-EFX	$1/2$-EF
★[1]	$3/4$-MMS	0.8253-MMS

For $n = 2$, EFX + ex ante EF was known [6], but not in polytime. We give an $O(m \log m)$-time algorithm [7]★.

References

[1] H. Akrami, J. Garg, ★E. Sharma, and S. Taki. Improving approximation guarantees for maximin share. In *EC*, 2024.

[2] H. Aziz, R. Freeman, N. Shah, and R. Vaish. Best of both worlds: Ex ante and ex post fairness in resource allocation. *Operations Research*, 2023.

[3] M. Babaioff, T. Ezra, and U. Feige. On best-of-both-worlds fair-share allocations. In *WINE*, 2022.

[4] I. Caragiannis, J. Garg, N. Rathi, ★E. Sharma, and G. Varricchio. New fairness concepts for allocating indivisible items. In *IJCAI*, 2023.

[5] U. Feige, A. Sapir, and L. Tauber. A tight negative example for MMS fair allocations. In *WINE*, 2021.

[6] M. Feldman, S. Mauras, V. V. Narayan, and T. Ponitka. Breaking the envy cycle: Best-of-both-worlds guarantees for subadditive valuations. In *EC*, 2024.

[7] J. Garg and ★E. Sharma. Best-of-both-worlds fairness of the envy-cycle-elimination algorithm. *arXiv:2410.08986*, 2024.

[8] J. Garg and S. Taki. An improved approximation algorithm for maximin shares. *Artificial Intelligence*, 2021.

[9] R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *EC*, 2004.